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FUNDAMENTALS OF FLUID DYNAMICS FOR AIRCRAFT DESIGNERS

By

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This Book Is Dedicated To

MR. ADOLPH LEWISOHN

PREFACE

During the twelve years that the author specialized on aerodynamics, many experiments were made to test various theories, and a considerable number of mathematical formulas were developed by him to explain phenomena as observed in research and flight tests. The results have appeared in more than seventy-five publications (listed in the Appendix of this volume). These many separate papers were written without a uniform viewpoint, and the earlier papers lacked the benefit of important data that developed later, consequently it now seems advisable to present the most useful portions of that extensive material for publication as a coordinated volume.

At this later date, it has been found practicable to simplify some of the earlier mathematical expressions; also, the progressive development for formulas, which was necessary to establish their value originally, is no longer essential where such formulas have since been accepted for general use. Furthermore, at this time it becomes practicable to devote more attention to the applications of the formulas rather than their derivation, together with comments on their relation to other theories; in fact that is the primary purpose of this book.

The short, simple formulas that are coming into use by aircraft designers can be employed to far better advantage by an understanding of the fundamental principles from which they have evolved. The manner of presenting herein the fundamentals of fluid dynamics as a digest of the theories that have withstood the test of time and trial, has in view the usefulness particularly for teachers, students of aeronautical engineering, and the designers of airplanes and airships.

Our knowledge of air motions is still very imperfect and chiefly the product of direct experience; even the results of

research experiments are often difficult to grasp. Nor is our theoretical knowledge really very wide However, there are a few simple relations that are exceedingly practical, and actually having so few, we should be eager to retain and use them. Because we are largely dependent on tests, there is even a greater need at this time for systematizing the immense amount of empirical information into a form suitable for persons engaged in this rapidly developing science. It is the author's earnest hope that this work will prove useful in disseminating a clearer conception of available knowledge relating to air motion in its application to flight phenomena.

It is pleasant duty to express thanks to Mr. Frederick N. Esher for most valuable assistance in pointing out errors in the manuscript, checking the mathematics, and providing the solutions to the problems in the text, as well as general suggestions leading to improvements in the book.

Washington, D. C., February 4, 1929. MAX M. MUNK.

CONTENTS

ILLUSTRATIONS

SYMBOLS USED IN THIS BOOK

A = constant factor.
a = distance.
B = constant factor.
b = span of wing.
C = constant factor.
C with subscript = coefficient, see subscript.
D = diameter, symbol of differentiation, drag.
C_D = drag coefficient.
c = chord of wing.
e = subscript effective.
F = function, potential function.
f with subscript = factor.
C_F = friction coefficient.
G = function.
g = subscript geometric.
H = function.
h = biplane gap.
i = subscript induced.
$i = \sqrt{-1}$.
I = moment of inertia.
K = volume of apparent mass.
k = volume factor of apparent mass.
L = lift, characteristic length.
C_L = lift coefficient.
M = moment or couple of forces.
M_m = momentum.
C_p = torque coefficient of propeller.
R = radius vector in polar coordinates.
S = area, wing area, airship cross-section.
s = length.
T = kinetic energy, thrust.
C_T = thrust coefficient.
U = velocity, tangential velocity.

u = velocity component.
V = velocity, velocity of flight.
v = velocity component.
w = velocity component.
x, y, z = space coordinates.
z = complex variable.

GREEK LETTERS

$\alpha, \beta, \delta, \epsilon$ = angles.
α = angle of attack.
η = efficiency.
μ = modulus of viscosity.
$\nu = \mu/\rho$ = modulus of kinematic viscosity.
ξ = ordinate of mean wing section.
ρ = mass density.
Φ = velocity potential.

FUNDAMENTALS
OF FLUID DYNAMICS
FOR AIRCRAFT DESIGNERS

INTRODUCTION

Below the maximum altitude where the atmosphere can be navigated, the air pressure varies between one-half and one atmosphere. The pressures produced by the motion of an aircraft are much smaller and only of the order of one-hundredth of one atmosphere. The density of air is proportional to its pressure. Hence, the density differences in the vicinity of the aircraft are likewise small when compared with the average air density. They influence the air motion much less than the friction forces between adjacent particles of air. No adequate analytic treatment of the larger friction effects is at present possible. It would then be unwise to complicate a general theory of the aeronautic air motions by including the effects of the compressibility of air. Accordingly, this theory definitely considers air as an incompressible fluid. Dynamics of fluids for aircraft designers constitutes throughout a special branch of hydrodynamics. The name aerodynamics suggests a science different from hydrodynamics and is in so far misleading.

It is customary to divide the remaining difficulties by temporarily disregarding the viscosity or internal friction of air, too. This, however, leads to serious discrepancies between the results obtained and the facts. It becomes necessary to modify the results and to bring them into better agreement with fact before applying them. This can only be done for certain classes of problems. These are the cases where the air flow is particularly regular and smooth. Small friction losses are a feature of such flows, and this is the reason why almost all flow problems arising in connection with aircraft design lend themselves to the procedure indicated.

A procedure fully analogous is followed in the study of motions of solids. There, too, the elasticity of the solids must be

definitely disregarded as being immaterial for the result. The subject is then likewise divided into theoretical dynamics, treating with frictionless solids and failing to yield correct results, and into a method for correcting the result, whenever that is possible. There are also criteria for the possibility of such correction, indicating whether the phenomenon is merely modified by friction or entirely governed by it. In hydrodynamics, unfortunately, reliable criteria of such kind are almost entirely lacking.

Furthermore, theoretical dynamics of solids forms the basis of theoretical hydrodynamics. The fluid is regarded as being composed of infinitely many infinitely small particles, the motion of each of them being governed by the fundamental laws found for the motion of solids. An intimate familiarity with these laws, though not a broad knowledge, is indispensable for the understanding of the principles of hydrodynamics. The student should be fully acquainted with what follows immediately.

With solids in equilibrium, all forces are occurring in pairs of two forces of equal magnitude acting along the same straight line, but in opposite direction. We arrive at single forces by the stratagem of introducing a fictitious closed surface, for instance, the surface of a solid. We consider now those forces only that are acting one way through the surface, for instance, only those acting on the solid from without. So, with a weight supported by a table, we consider the gravity force exerted on the weight, but not the one exerted by the weight. Further, we consider the support of the table, but not the pressure on the table. The weight is then in equilibrium if and only if the forces considered possess a vanishing resultant.

We proceed now to dynamics, that is, to the absence of equilibrium. The resultant of all forces acting one way through the fictitious surface does not vanish now, but rather becomes manifest by accelerating the solid in the well-known way. This very simple relation unfortunately has been somewhat obscured by

the invention of the mass forces. The latter are merely fictitious, and by their use the phraseology of statics (theory of equilibrium) can be employed for dynamical problems (without equilibrium). A fictitious equilibrium is established where there actually is none. The acceleration consumes the resultant force; the mass force would neutralize it. Consider a hammer whirled around on a string. The tension of the string is by no means neutralized by the centrifugal force of the hammer, but the former actually comes into effect and causes the hammer to deviate from the straight path.

Referring to the textbooks on mechanics for further elucidation on dynamics of solids, we enter at once into our particular subject.

CHAPTER I

THE CLASSICAL PRINCIPLES OF HYDRODYNAMICS

1. Outlook

We shall discuss the principles of hydrodynamics only so far as is necessary for the understanding of the applications we are going to make. We shall see how the invention of the velocity potential relieves the investigator from the consideration of all forces, pressures, and accelerations throughout the fluid, and how the potential can be determined to represent a motion of an incompressible fluid.

This potential again is only the representative of a pressure that may have put the fluid into its motion, and it stands for the fact that pressures have actually put it into motion. Each small particle of the fluid, floating in the remainder of the fluid, experiences a buoyancy in accordance with Archimedes' law, which accelerates it but does not change its moment of momentum with respect to its own center of gravity. This moment of momentum, then, remains zero at all times, if it was so at the beginning. The potential expresses this absence of the particles' own moment of momentum, of their rotation as it is called.

The mathematical expression of these principles will bring them out more clearly than common language.

2. Notations

We have first to review the common notations and to introduce a simplifying way of writing derivatives. Let x, y, and z denote the Cartesian space coordinates at right angles to each other, and u, v, and w the velocity components parallel to them. Let t denote the time, p the pressure, and ρ the mass of one unit of volume of the fluid. We shall have to consider two different

kinds of derivatives. In the first we fix attention to one particular point in space, paying no regard to the fact that its place is occupied by various fluid particles one after the other. We merely observe the changes going on at the considered point and the differences of the value of quantities at that point and at others near by. The derivatives resulting from such attitude of observation, considering the space ordinates and time as independent variables, are called local derivatives, as for instance, $\partial v/\partial x$; and we are going to write them v_x, etc. The symbol v_t or $\partial v/\partial t$ is not the component of acceleration of the particle, but the rate of change of the velocity at a certain point (not of a certain particle).

The other way is fixing our attention to individual fluid particles, following them as they flow along. The rates of change occurring for individual particles are called absolute derivatives, and may be denoted by using a capital D as differentiation symbol, thus Dv/Dt which expression is the acceleration component of the particle in the direction of the y axis.

We introduce at last the so-called "rotation" of each particle. This is twice the value of its mean angular velocity, or better said, twice the ratio of its moment of momentum to its moment of inertia, both with respect to its own center of gravity. The components of the rotation are accordingly

$$u_y - v_x; \quad v_z - w_y; \quad \text{and } w_x - u_z.$$

u_y is indeed the angular velocity of a line element made of particles parallel to the y axis, and $-v_x$ the same, parallel to the x axis. Their sum can, therefore, be said to represent twice the average component of the angular velocity of the particle.

The magnitude of the rotation, so computed, does not depend on the choice of the direction of the axes of coordinates. This follows from the assumption that the differences of the velocity components at different points approach proportionally to the differences of the coordinates of these points, as they become small. This assumption, without being particularly true, is an

idealization particularly practical, and in some form is the basis of most mathematical investigations of physical problems. The mathematics of the proof that the one follows from the other is simple and somewhat trivial and out of line with our major development. It may, therefore, be left to the reader.

With the above notations we have the relation

$$Du/Dt = u_t + u\,u_x + v\,u_y + w\,u_z \tag{1}$$

and two similar relations for Dv/Dt and Dw/Dt. This relation expresses that the acceleration, the rate of change of velocity of one individual particle, is made up of two parts. The first part is the time-rate of change of the velocity at the particular point in question. The other part is the rate of change the particle experiences from traveling into a region of higher (or lower) velocity. In the y direction, it advances by the distance v per unit of time. The local rate of change per unit distance is u_y. Hence, the product, $v\,u_y$ is the rate of change per unit of time, and similarly with the other terms.

We dismiss gravity from the very first. Its effect is counterbalanced by the well-known upward pressure decrease of the atmosphere. We rather measure all pressures relative to the pressure of the air at rest at that point, and by so doing save ourselves the trouble of dealing with any external forces acting on the air. The only remaining force to which each particle is subjected is the effect of the pressure. This force is zero if the pressure is equal at all points. It is generally variable, however, and then each particle experiences a buoyancy force equal to its volume multiplied by the so-called gradient of the pressure. The components of this gradient are $-p_x$, $-p_y$, and $-p_z$, respectively. This expression for the buoyancy can easily be obtained from the contemplation of a small box with the sides dx, dy, and dz. $-dx\,dy\,dz\,p_x$ is directly seen to be the excess force on the two outer surfaces with the sides dy and dz. This is, therefore, the component of the buoyancy at right angles to the two faces.

3. Absence of Rotation

After this list of notations, definitions, and trivial transformation equations resulting therefrom, we proceed to prove that the rotation remains zero once it was so. This absence of rotation will prove an enormous simplification of the mathematical treatment of our problems. We start by applying Newton's law to the acceleration of one fluid particle and the force acting on the particle.

$$Du/Dt = u_t + u\,u_x + v\,u_y + w\,u_z = -\,p_x/\rho \tag{2}$$

and two similar equations for the y and z direction.

We next write down the absolute rate of change of the rotation component $u_y - v_x$, expecting to find it to be zero.

$$D(u_y - v_x)/Dt = \partial(u_y - v_x)/\partial t + u\,\partial(u_y - v_x)/\partial x + v\,\partial(u_y - v_x)/\partial y + w\,\partial(u_y - v_x)/\partial z \tag{3}$$

We assume $(u_y - v_x)$ to be zero at all points at one moment, and examine whether it will remain zero at the next moment. This assumption makes the last three terms of equation (3) zero and it remains only

$$D(u_y - v_x)/Dt = \partial(u_y - v_x)/\partial t = \partial u_t/\partial y - \partial v_t/\partial x \tag{4}$$

We transform now the right-hand side of equation (2) by making use of $u_y = v_x$, which is equivalent to $u_y - v_x = 0$. We obtain

$$Du/Dt = u_t + u\,u_x + v\,v_x + w\,w_x = u_t + \tfrac{1}{2}(u^2 + v^2 + w^2)_x = -\,p_x/\rho \tag{5}$$

$$u_t = -\,p_x/\rho - \tfrac{1}{2}(u^2 + v^2 + w^2)_x \tag{6}$$

and in the same way

$$v_t = -\,p_y/\rho - \tfrac{1}{2}(u^2 + v^2 + w^2)_y$$

Hence, substituting this into (4), we obtain finally

$$D(u_y - v_x)/Dt =$$

$$-\left(\frac{p}{\rho} + \tfrac{1}{2}(u^2 + v^2 + w^2)\right)_{xy} + \left(\frac{p}{\rho} + \tfrac{1}{2}(u^2 + v^2 + w^2)\right)_{xy} = 0 \tag{7}$$

That would ordinarily prove the proposition. This proof is a specialization of the proof of a more general theorem about the rotation, as it was originally given by Helmholtz, and shows how the proposition was obtained. The rotation is assumed to be zero at all points, but at the present time only. Its rate of change follows then to be zero. From a very rigid mathematical viewpoint this now does not prove the absence of rotation but is a necessary condition only. We will confine ourselves to this most important part of the proof, which is too abstract already for the aim of this book, and refer the mathematically interested reader to the treatises on hydrodynamics. We others accept the theorem on the credit of the many great mathematicians who examined it and found it correct.

4. Velocity Potential

We merely integrate equation (6) with respect to t,

$$u = -\partial/\partial x \int \left(\frac{u^2 + v^2 + w^2}{2} + p/\rho\right) d\,t \tag{8}$$

The expressions for v and w corresponding to equation (8) are formed by replacing x by y or z, since the integral itself is symmetrical with respect to the three coordinates.

We see now from equation (8) that there is a quantity existing throughout the flow, different in general from point to point and from time to time, of which the velocity is the gradient. The velocity is a gradient. That is something we could not see beforehand, for not every velocity distribution can be represented as a gradient. On the contrary, the velocity distribution of all points of a turning flywheel, for instance, cannot. Also, the viscosity of air causes the actual flow to deviate slightly at least from a flow that can be represented as a gradient. But under the assumptions made, velocity is actually a gradient of some other quantity. This quantity, the integral in equation (8), is called velocity potential in analogy to the gravity potential and the magnetic and electrical potential, which require the same mathe-

matical treatment as the velocity potential. It will be denoted by Φ.

The air flows of the aeronautic problems are then found to be potential flows, and conversely, the existence of a potential of a flow specifies the flow as complying with some of the assumptions made for these flows.

5. Physical Meaning of the Velocity Potential

The physical interpretation of the velocity potential is very important for the understanding of its applications. We suppose the air flow to be created from rest by a constant pressure distribution existing during a short time interval dt only. This pressure must be very large, accordingly, and hence the first term in the integral (8) can then be neglected when compared with p/ρ. This integral appears to be an impulsive pressure, analogous to the impulse of a hammer blow, say, that is not measured by the magnitude of the exerted force, but by the time integral of this force. The velocity potential is accordingly seen to be the quotient of the density into the impulsive pressure distribution necessary to bring the flow to a stop. Regarding such interpretation of the potential, as standing for the creating impulsive pressure, will make self-evident the chief relations we have to use.

6. Air Pressure

For $u^2 + v^2 + w^2$, we can write V^2 where V is the magnitude of the velocity. Equation (6) therefore can be written

$$-p_x/\rho = u_t + \tfrac{1}{2}(V^2)_x \tag{9}$$

Integrating this equation with respect to dx, we obtain

$$-p = V^2\rho/2 + \rho\frac{\partial}{\partial t}\int u\,dx = V^2\rho/2 + \rho\Phi_t + \text{const.} \tag{10}$$

This is an important and very simple relation between the pressure, the velocity, and the rate of change of the potential. Most problems of the aircraft designer deal with steady flows.

That means the conditions remain always the same at each particular point, and local derivatives with respect to time are zero. Equation (10) becomes then even simpler.

$$p + V^2\rho/2 = \text{constant} \tag{11}$$

This expresses Bernouilli's theorem about the pressure. It says, that in a steady potential flow, the pressure throughout the fluid depends on the velocity only, under the assumptions made. The points of large pressure are the ones of small velocity. The points of maximum pressure are the points of the velocity zero.

This very simple relation admits of direct proofs in special cases. Along one streamline it can be directly deduced from the momentum theorem or from the energy theorem, For, the pressure is the transported energy per unit of delivered volume. The relation is easiest seen with water flowing out of a small aperture of a container which is filled to a height h over that opening. The kinetic energy of the water is then equal to the work performed by gravity through the height h. The pressure height at the level of the aperture is equal to the pressure of a water column of the height h. The velocity at last is equal to the velocity of free fall corresponding to the same height. This gives exactly Bernouilli's relation, and it must be realized that the pressure of the mentioned magnitude only exists at points distant from the aperture, where the velocity of flow is practically zero. In the jet itself is atmospheric pressure.

Bernouilli's theorem expresses more than that the sum of equation (11) is constant along each streamline. It says that under the assumptions made, the constant of this equation is the same throughout the entire flow.

In view of the importance of Bernouilli's equation (5), its left side has received a name and is called "total pressure." The term $V^2\rho/2$ is called "dynamic pressure," being the part of the pressure arising from dynamic effects. The pressure p itself is then called "static pressure" to distinguish it from the dynamic pressure.

This must not be misunderstood. Of course, at each point, there is only one pressure existing, p, the static pressure. Each particle is under the action of the adjacent particles, measured by the static pressure. The dynamic pressure is something obtained by reasoning and of fictitious existence only.

Some confusion has arisen about this plain relation from the fact that certain methods of measurement yield the total pressure rather than the static pressure. That strengthens a misconceived notion about the reality of the total pressure. It is true, a measured pressure is actually existing after the proper instrument has been inserted into the air flow, but it may not have existed before the instrument was inserted, and our statements refer to pressures in absence of any inserted instruments. A great variety of pressures can indeed be obtained by inserting various instruments, but none of these pressures should be assigned to the undisturbed flow.

7. Condition of Incompressibility

The flows of incompressible and of compressible fluids may equally well possess a potential. It remains to express the mathematical limitation to the potential expressing that the velocity distribution represented by it is the velocity distribution of an incompressible fluid. The condition of incompressibility as such is again independent of the existence of the velocity potential, it is independent of any dynamic relation and holds, therefore, equally for viscous and unviscous fluids. We proceed to express in mathematical terms the purely kinematic condition of incompressibility with the intention to combine it later with the dynamic conditions, which means to apply it to the potential.

Only one particle can occupy a place at a time, and on the other hand, no vacant spaces occur. This physical condition is equivalent with the condition that during a short-time interval equal volumes of fluid are entering and leaving a small box with the sides dx, dy, and dz. These latter denote again the entials of the Cartesian space coordinates x, y, and z at

angles to each other. Denoting again by u, v, and w the velocity components parallel to x, y, and z, u_x is the average difference of the velocity components normal to the two opposite faces at right angles to the x axis, hence $u_x\, dx\, dy\, dz$ is the volume of the fluid leaving the box through one of these two faces in excess to the volume entering through the other face, all per unit of time. Hence $u_x + v_y + w_z$ is the entire excess flow per unit volume and per unit time. Our kinematic condition is expressed by putting this "divergence" zero. The equation of continuity for an incompressible fluid is accordingly

$$u_x + v_y + w_z = \partial u/\partial x + \partial v/\partial y + \partial w/\partial z = 0 \qquad (12)$$

It is called equation of continuity because it expresses that the space is continuously filled at all times. Equation (12) is the desired expression for the kinematic condition of incompressibility. If a velocity potential exists, the velocity components can be expressed:

$$u = \Phi_x; \quad v = \Phi_y; \quad \text{and } w = \Phi_z.$$

Substituting this into equation (12) gives the combined expression

$$\partial^2\Phi/\partial x^2 + \partial^2\Phi/\partial y^2 + \partial^2\Phi/\partial z^2 = 0 \qquad (13)$$

called Laplace's equation.

Any flow having a potential complying with Laplace's equation is in keeping with both our kinematic and our dynamic conditions. No further check of any kind is needed. The flow is entirely determined by equation (13) together with the special kinematic conditions on the surface of the solids creating the flow. This latter means that the air must move in such a way as to accommodate at all times the moving solids.

This concludes the derivation of our two fundamental equations, (10) and (13). Their application will become clearer as our work progresses. Their meaning too, and some interposed remarks on Laplace's equation may prove profitable as a preparation for the later application.

Laplace's equation is a partial differential equation, for it contains derivatives with respect to more than one variable. The time does not occur at all, but still Φ may vary with time too. The absence of t in the equation merely indicates an unsteady flow to consist successively of flows each of which might exist as a steady flow.

With partial differential equations, now, the problem to find the general solution subordinates itself to the problem how to specialize the solution for the boundary conditions in question, in our case for the conditions on the surface of the solids. We shall restrict our investigation to very simple surfaces, chiefly to the straight line in a two-dimensional flow. This line is the representative of a plane strip with the flow equal in all planes at right angles to it and no velocity component parallel to it. Our solution will appear as an infinite series, each term of which being by itself a solution of the differential equation. This is possible because the sum of any solutions of equation (13) each multiplied by any constant is again a solution, as can easily be seen by substituting such sums into equation (13). All we do is to write down an expression that satisfies the differential equation (13) at all points except at the points of the straight line. That expression contains a parameter and is either found in the mathematical literature, or worked out by intuition, analogy, and trial. Inserting consecutive integers for the parameters, multiplying each time with a constant, and summing up all products gives then the solution, and we shall see in our case that the expression is of such nature as to make it easy to find the correct constants.

8. Kinetic Energy, Momentum and Apparent Mass

We return to the consideration of the flow and finish the list of principal conceptions we are going to use. The impulsive pressure necessary to create the flow and represented by the velocity potential is most directly connected with the main conceptions referring to such flows. The resultant of this im-

pulsive pressure over the surface of the solid creating the flow is an impulsive force, or an impulsive couple. These are forces or couples of very large magnitude acting during a very short time interval, measured by their time integral. During the short creation of the flow the solid is moving through a short path. Hence, mechanical work is performed. This work is transferred to the fluid and is stored there as kinetic energy of flow. The fluid can also be said to have absorbed the impulsive force and to contain it as momentum of the flow, or the impulsive couple and contain it as moment of momentum.

This leads directly to expressions giving the kinetic energy and the momentum of the flow in terms of its potential. Let dn be a linear element at right angles to the surface of the solid drawn outward. The final velocity component of the flow at right angles to the surface is then $-\partial\Phi/\partial n$, and the impulsive pressure $-\rho\Phi$ acts through the distance $-\frac{1}{2}(\partial\Phi/\partial n)dt$, in creating the flow.

The work performed all over the surface is, therefore,

$$T = \int \frac{\rho}{2}\Phi\frac{\partial\Phi}{\partial n}dS \tag{14}$$

which integral is to be extended over the entire surface of the solid consisting of all surface elements dS. The expression under the integral contains the mass of the element of fluid displaced by the surface element per unit of time, each element of mass multiplied by the velocity potential.

The component of the momentum in any direction results in a similar way to be directly equal to

$$M = -\int \rho\Phi dS' \tag{15}$$

that is, to the integral of the potential with respect to the projections of the surface elements in the direction of the component—multiplied by the density. It is often more convenient to compute the component of momentum, parallel to the motion of the solid, from the kinetic energy (14). The equation is then

$$T = \tfrac{1}{2}VM$$

the kinetic energy is half the product of the momentum and of the path through which it is produced. The analogy between these simple relations and the equations for the motion of one solid particle is carried on further by the introduction of the "apparent mass." The potential, and hence also its derivative in (14), is directly proportional to the velocity at which the solid is moving. Hence, the kinetic energy is proportional to the square of the velocity; it is the product of this square by the density and by an expression depending on the geometric dimensions of the solid only, the expression of a certain volume. The effect of the fluid surrounding the solid is, therefore, fully described by assigning to the solid a mass in addition to its original or actual mass. This mass is properly called additional apparent mass, and is the mass of the additional apparent volume if filled with the fluid by which the solid is surrounded. This apparent volume may be denoted by K, its magnitude is

$$K = T/(V^2\rho/2) \tag{16}$$

There is, however, one important difference between the mass of a particle and the apparent mass of a solid. The former is the same for all directions of motion, the latter is not necessarily. This follows from the fact that the momentum of flow is not necessarily parallel to the direction of motion of the solid giving rise to the flow. This complicates the kinematics of a solid in a perfect fluid somewhat, and makes it in some ways similar to the kinematics of a gyroscope, which latter possesses, in general, different moments of inertia with respect to different axes.

9. Synopsis

The air flows in aeronautics have a potential, complying with Laplace's equation.

The solution of a problem requires the determination of this potential such that the represented flow is in keeping with the motion of the solids.

The velocity at each point is the gradient of this potential. The pressure at each point is given by Bernouilli's equation, if the flow is steady; otherwise an additional term must be added. The air forces are computed from the pressures by integration. The use of the kinetic energy of the flow, its resultant momentum, and the apparent mass of the solid will save useless repetitions of some of these steps, thereby simplifying the solution.

10. Problems and Suggestions

1. Examine whether the potential $-A/\sqrt{x^2+y^2+z^2}$ of the point source complies with Laplace's equation. How does the velocity vary along the line $y = 0$, $z = 0$?

2. The velocity components of a two-dimensional flow are

$$u = \frac{Ay}{x^2+y^2}; \qquad v = \frac{-Ax}{x^2+y^2}.$$

Is this a flow of constant density? Is it a flow of zero rotation?

3. The velocity potential $Ax + By + Cz + D$ represents a parallel flow of constant velocity, independent of the value of D. How large is the velocity?

4. The derivative Φ_x of a solution Φ of Laplace's equation is again a solution.

5. The magnitude of the velocity component u of a potential flow of constant density is the velocity potential of another flow of constant density.

6. Apply the last theorem to the potential in Problem 1, and describe the resulting flow.

7. $\Phi = \dfrac{Ax}{x^2+y^2+z^2} + Bx$ represents the flow resulting when a sphere of the radius $\sqrt{A/B}$ is placed in a parallel flow.

8. Compute the dynamic pressure corresponding to the velocity of flight of 120 ft./sec. in air of a density 1/420 lb. sec^2 per ft^4.

9. In Problem 7 the velocity at a large distance from the sphere is B. The maximum velocity at the surface of the sphere

is at the points $x = 0$; $y^2 + z^2 = A/B$. How large is this maximum velocity? Its ratio to the velocity of flow at a large distance is equal for all spheres and velocities.

10. Compute the lowest pressure at the surface of the sphere in the last problem for the velocity of flow 60 ft./sec. and the density of air 1/420 lb. sec^2/ft^4.

11. $\Phi = - x/(x^2 + y^2)$ represents the two-dimensional flow created by a circular cylinder with the radius 1 moving with the velocity 1. The radial velocity component at the surface of the cylinder $x^2 + y^2 = 1$ is x. The kinetic energy per unit length of the cylinder of flow outside the cylinder is $\frac{\rho}{2}\int x^2 ds$, this integral to be taken around the circle $x^2 + y^2 = 1$ with the elements of periphery ds. Hence, the volume of apparent additional mass of the cylinder is equal to its volume.

12. The apparent additional mass of a sphere is half its volume. A sphere of specific gravity 2 is falling within a perfect fluid of the specific gravity 1. How large is its acceleration of fall?

CHAPTER II

THE AERODYNAMIC FORCES ON AIRSHIP HULLS

11. Outlook

It is to the airship hull as a solid creating a pure potential flow that the chief application of hydrodynamics in its classical state is made. We shall express all air forces and their distribution by means of the apparent mass of the solid.

At first we consider the hull as a whole and find the resultant momentum to be directly proportional to the transverse component of the momentum of the flow, and this in turn directly proportional to the difference of the apparent masses of the hull in the two main directions. The relation is then generalized to include the motion along a circular path.

We next assume the hull to be very elongated, so that the transverse flow becomes two-dimensional. This leads to a simple formula about the distribution of the transverse air force, containing the apparent mass of the cross-section, the area of which is equal to the cross-section itself in case of a circular cross-section.

The same two-dimensional flow is used for deriving the pressure distribution around the cross-section not too close to the bow. The pressure distribution over the bow is discussed through the pressure distribution over an ellipsoid, again expressed by means of its apparent masses.

Thus all questions are covered. The chapter is completed by remarks referring to the application of the theory.

12. Resultant Airship Force in Straight Motion

Airship hulls moving through the air give rise to an air flow well approximated by a potential flow. There may be a large

resultant moment of the air forces, but only a comparatively small lift and drag. The bulk of the lift of airship hulls is produced by the fins. We can, therefore, successfully apply the theory of potential flows to the investigation of the air flow around airship hulls. With wings the conditions are different as there is a considerable lift. Therefore, the discussion of the chapter does not directly apply to wings.

We proceed to express the resultant moment by means of the volumes of apparent mass of the hull and have to begin with the discussion of the momentum of the flow. This resultant momentum is generally not parallel to the direction of motion of the solid. There are, however, always at least three axes of the solid, mutually at right angles to each other, so that motion and momentum are parallel if these axes are turned into the direction of motion. For airship hulls this follows directly from their symmetry. We dispense, therefore, with the general proof; it can be based on the geometric proposition that all quadrics have at least three axes of symmetry mutually at right angles to each other.

Let now the solid proceed in any orientation with respect to the direction of motion, and divide the velocity of flight into three components parallel to the main axes mentioned before. Let these components of the velocity of flight be u, v, and w. Let further K_1, K_2, and K_3 denote the volumes of apparent mass of the solid when moving parallel to the main axes. Then $\rho u K_1$, $\rho v K_2$, and $\rho w K_3$ are the resultant momenta for the solid moving parallel to its main directions. These are also the components of the momentum for the general motion having the components u, v, and w. Otherwise expressed, the momentum can be built up of components just as the velocity can. This follows directly from the superposition theorem of the potential and from the definition of the momentum.

The motion may now take place at right angles to one of the axes. One component of the momentum is then zero, and the momentum is in the plane of the two main axes of motion. Its

components are $\rho u K_1$ and $\rho v K_2$. The component of this momentum at right angles to the motion is then found by the rules of geometry:

$$M_t = \frac{\rho u v}{\sqrt{u^2 + v^2}} (K_2 - K_1)$$

Introducing the angle α between the direction of motion and one axis of the solid, this can be written:

$$M_t = (V\rho/2)\,(K_2 - K_1)\,\sin 2\alpha \qquad (1)$$

We have thus derived the magnitude of the transverse component of the momentum, that is to say, the component at right angles to the direction of flight. That is almost all that is necessary for the derivation of the final result. It can easily be seen that in case of steady motion the desired moment of the air pressures is directly proportional to the transverse momentum of the flow. The moment of this momentum with respect to a point fixed in space increases continually by the product of the momentum and the path traveled, and an equivalent force must act on the hull. While the hull is proceeding, the transverse momentum at one point is continually annihilated and replaced by another of the same magnitude and direction placed further ahead so as to keep its position relative to the hull. This requires a couple equal to the product of the transverse momentum and the velocity of flight. Hence, it appears that an airship hull, flying steadily under the angle of attack α with the velocity of flight V experiences a resultant couple of the magnitude

$$M = (V^2 \rho/2)\,(K_2 - K_1)\,\sin 2\alpha \qquad (2)$$

K_1 and K_2 denote the volumes of apparent mass with respect to the longitudinal and transverse main axes of the hull. This moment is unstable; it tends to increase the angle of attack. The bow, when turned up, presses the air down and hence is itself lifted up. Each hull requires, therefore, fins for stabilization.

Airship hulls are often bounded by surfaces of revolution. In addition, they are usually rather elongated, and if the cross-sections are not exactly round, they are at least approximately of equal and symmetrical shape and arranged along a straight axis. Surfaces of revolution have, of course, equal transverse apparent masses; each transverse axis at right angles to the axis of revolution is a main direction. For very elongated surfaces of revolution a further important statement may be made regarding the magnitude of the longitudinal and transverse apparent mass. When moving transversely the flow is approximately two-dimensional along the greatest part of the length. The apparent additional mass of a circular cylinder moving at right angles to its axis will be shown to be equal to the mass of the displaced fluid. It follows, therefore, that the apparent transverse additional mass of a very elongated body of revolution is approximately equal to the mass of the displaced fluid. It is slightly smaller, as near the ends the fluid has opportunity to pass the bow and stern. For cross-sections other than circular, the two main apparent masses follow in a similar way from the apparent mass of the cross-section in the two-dimensional flow.

The longitudinal apparent additional mass, on the other hand, is small when compared with the mass of the displaced fluid. It can be neglected if the body is very elongated or can at least be rated as a small correction. This follows from the fact that only near the bow and the stern does the air have velocities of the same order of magnitude as the velocity of motion. Along the ship the velocity not only is much smaller but its direction is essentially opposite to the direction of motion, for the bow is continually displacing fluid and the stern makes space free for the reception of the same quantity of fluid. Hence, the fluid is flowing from the bow to the stern, and as only a comparatively small volume is displaced per unit of time and the space is free in all directions to distribute the flow, the average velocity will be small.

It is possible to study this flow more closely and to prove analytically that the ratio of the apparent mass to the displaced mass approaches zero with increasing elongation. This proof, however, requires the study or knowledge of quite a number of conceptions and theorems, and it seems hardly worth while to go through all this in order to prove such a plausible and trivial fact for the general case.

The actual magnitudes of the longitudinal and transverse masses of elongated surfaces of revolution can be studied by means of exact computations made by H. Lamb with ellipsoids of revolution of different ratios of elongation. The figures of k_1 and k_2, where $K = k \times$ Volume, obtained by him are contained in Table I;[1] $k_2 - k_1$ is computed. For bodies of a shape reasonably similar to ellipsoids it can be approximately assumed that $(k_2 - k_1)$ has the same value as for an ellipsoid of same length and volume; i.e., if Vol./L^3 has the same value.

13. Resultant Air Force in a Circular Path

The next problem of interest is the resultant aerodynamic force if the hull rotates with constant velocity around an axis outside of itself. That is now comparatively simple, as the arguments of the last section have only to be repeated. The configuration of flow follows the body, with constant shape, magnitude, and hence, with constant kinetic energy. The resultant aerodynamic force, therefore, must be such as neither to consume nor to perform mechanical work. This leads to the conclusion that the resultant force must pass through the axis of rotation. In general, it has both a component at right angles and one parallel to the motion of the center of the body.

We confine the investigation to a surface of revolution. Let an airship with the apparent masses $K_1\rho$ and $K_2\rho$ and the apparent moment of inertia $K'\rho$ for rotation about a transverse axis through its aerodynamic center move with the velocity V of its aerodynamic center around an axis at the distance r from its

[1]See page 85.

aerodynamic center and let the angle of yaw φ be measured between the axis of the ship and the tangent of the circular path at the aerodynamic center. The ship is then rotating with the constant angular velocity V/r. The entire motion can be obtained by superposition of a longitudinal motion $V \cos \varphi$ of the aerodynamic center of the hull, a transverse velocity $V \sin \varphi$, and an angular velocity V/r. The longitudinal component of the momentum is $V\rho \cos \varphi\, k_1 \cdot$ Volume, and the transverse component of the momentum is $V\rho \sin \varphi\, k_2 \cdot$ Volume. Besides, there is a moment of momentum due to the rotation. This can be expressed by introducing the apparent moment of inertia $K'\rho = k'J\rho$, where J is the moment of inertia of the displaced air; thus making the angular momentum

$$k' J\rho V/r$$

As it does not change, it does not give rise to any resultant aerodynamic force or moment during the motion under consideration.

The momentum remains constant, too, but changes its direction with the angular velocity V/r. This requires a force passing through the center of turn and having the transverse component

$$F_t = K_1 \rho \cos \varphi\, V^2/r$$

and the longitudinal component

$$F_l = K_2\, \rho \sin \varphi\, V^2/r$$

The first term is some kind of centrifugal force. Some air accompanies the ship, increasing its longitudinal mass and hence its centrifugal force. It will be noticed that with actual airships this additional centrifugal force is small, as k_1 is small.

The force attacking at the center of the turn can be replaced by the same force attacking at the aerodynamic center and a moment around this center of the magnitude

$$M = (K_2 - K_1) \sin 2\varphi\, V^2 \rho/2 \qquad (2)$$

This moment is equal in direction and magnitude to the unstable

moment found during straight motion under the same angle of pitch or yaw. The longitudinal force is in practice a negative drag as the bow of the ship is turned toward the inside of the circle. It is of no great practical importance as it does not produce considerable structural stresses.

It appears thus that the ship when flying in a curve or circle, experiences almost the same resultant moment as when flying straight and under the same angle of pitch or yaw. I proceed to show, however, that the transverse aerodynamic forces producing this resultant moment are distributed differently along the axis of the ship in the two cases.

14. Distribution of the Forces Along the Axis

The distribution of the transverse aerodynamic forces along the axis can conveniently be computed for very elongated airships. It may be supposed that the cross-section is circular, although it is easy to generalize the proceeding for a more general shape of the cross-section.

The following investigation requires the knowledge of the apparent additional mass of a circular cylinder moving in a two-dimensional flow. I proceed to show that this apparent additional mass is exactly equal to the mass of the fluid displaced by the cylinder.

In the two-dimensional flow the cylinder is represented by a circle. Let the center of this circle coincide with the origin of a system of polar coordinates R and φ, moving with it, and let the radius of the circle be denoted by r. Then the velocity potential of the flow created by this circle moving in the direction $\varphi = 0$ with the velocity v is $\Phi = vr^2 \cos \varphi/R$. For, this potential gives the radial velocity component

$$\partial\Phi/\partial R = -v(r^2/R^2) \cos \varphi$$

and at the circumference of the circle this velocity becomes $v \cos \varphi$, and this is, in fact, the normal component of velocity of a circle moving with the velocity v in the specified direction.

The kinetic energy of this flow is now to be determined. In analogy to equation (8), Section 4, this is done by integrating along the circumference of the circle the product of (a) the elements of half the mass of the fluid penetrating the circle ($\rho/2 \cdot \cos \varphi \; vr \, d\varphi$), and (b) the value of the velocity potential at that point ($- v \cos \varphi \; r$). The integral is, therefore,

$$\frac{\rho}{2}\int_0^{2\pi} \cos^2\varphi \quad v^2 r^2 d\varphi$$

giving the kinetic energy

$$r^2 \pi v^2 \quad \rho/2$$

This shows, in fact, that the area of apparent mass is equal to the area of the circle. We are now enabled to return to the airship.

If a very elongated airship is in translatory horizontal motion through air otherwise at rest and is slightly pitched, the component of the motion of the air in the direction of the axis of the ship can be neglected. The air gives way to the passing ship by flowing around the axis of the ship, not by flowing along it. The air located in a vertical plane at right angles to the motion remains in that plane, so that the motion in each plane can be considered to be two-dimensional. Consider one such approximately vertical layer of air at right angles to the axis while the ship is passing horizontally through it. The ship displaces a circular portion of this layer, and this portion changes its position and its size. The rate of change of position is expressed by an apparent velocity of this circular portion, the motion of the air in the vertical layer is described by the two-dimensional flow produced by a circle moving with the same velocity. The momentum of this flow is $Sv\rho dx$, where S is the area of the circle, and v the vertical velocity of the circle, and dx the thickness of the layer. Consider first the straight flight of the ship under the angle of pitch φ. The velocity v of the displaced circular portion of the layer is then constant over the whole length of the ship and is $V \sin \varphi$, where V is the velocity of the airship

along the circle. Not so the area S; it changes along the ship. At a particular layer it has the rate of change per unit time,

$$V \quad \cos\varphi \quad dS/dx$$

where x denotes the longitudinal coordinate.

Therefore, the momentum has the rate of change

$$V^2\rho/2 \quad \sin 2\varphi \quad (dS/dx)dx$$

This gives a down force on the ship with the magnitude

$$dF = dx \quad V^2\rho/2 \quad \sin 2\varphi \quad dS/dx \tag{3}$$

Next, consider the ship when turning, the angle of yaw being φ. The momentum in each layer is again

$$vS\rho dx$$

The transverse velocity v is now variable, too, as it is composed of the constant portion $V \sin \varphi$, produced by the yaw, and of the variable portion $Vx/r \cos \varphi$, produced by the turning. $x = 0$ represents the aerodynamic center. Hence, the rate of change of the momentum per unit length is

$$V^2\frac{\rho}{2} \quad \sin 2\varphi \quad \frac{dS}{dx} + \rho\frac{V^2}{r} \quad \cos^2\varphi \quad \frac{d}{dx}(xS)$$

giving rise to the transverse force per unit length

$$V^2\frac{\rho}{2} \quad \sin 2\varphi \quad \frac{dS}{dx} + \rho\frac{V^2}{r} \quad \cos^2\varphi \quad \left(S + x\frac{dS}{dx}\right)$$

or otherwise written

$$dF = dx\left(V^2\frac{\rho}{2} \quad \sin 2\varphi \quad \frac{dS}{dx} + V^2\frac{\rho}{r} \quad \cos^2\varphi \quad S + V^2\frac{\rho}{r} \quad \cos^2\varphi x\frac{dS}{dx}\right) \tag{4}$$

The first term agrees with the moment of the ship flying straight having a pitch φ. The direction of this transverse force is opposite at the two ends, and gives rise to an unstable moment. The ships in practice have the bow turned inward when they fly in turn. Then the transverse force represented by the first term of equation (4) is directed inward near the bow and outward near the stern.

The sum of the second and third terms of equation (4) gives no resultant force or moment. The second term alone gives a transverse force, being in magnitude and distribution almost equal to the transverse component of the centrifugal force of the displaced air, but reversed. This latter becomes clear at the cylindrical portion of the ship, where the two other terms are zero. The front part of the cylindrical portion moves toward the center of the turn and the rear part moves away from it. The inward momentum of the flow must change into an outward momentum, requiring an outward force acting on the air, and giving rise to an inward force reacting this change of momentum.

The third term of (4) represents forces almost concentrated near the two ends and their sum in magnitude and direction is equal to the transverse component of the centrifugal force of the displaced air. They are directed outward.

Ships only moderately elongated have resultant forces and a distribution of them differing from those given by the formulas (3) and (4). The assumption of the layers remaining plane is more accurate near the middle of the ship than near the ends, and in consequence the transverse forces are diminished to a greater extent at the ends than near the cylindrical part when compared with the very elongated hulls. In practice, however, it will often be exact enough to assume the same shape of distribution for each term and to modify the transverse forces by constant diminishing factors. These factors are logically to be chosen differently for the different terms of (4). For the first term represents the forces giving the resultant moment proportional to $(k_2 - k_1)$, and hence it is reasonable to diminish this term by multiplying it by $(k_2 - k_1)$. The second and third terms take care of the momenta of the air flowing transversely with a velocity proportional to the distance from the aerodynamic center. The moment of inertia of the momenta really comes in, and therefore it seems reasonable to diminish these terms by the factor k', the ratio of the apparent moment of inertia to the moment of inertia of the displaced air.

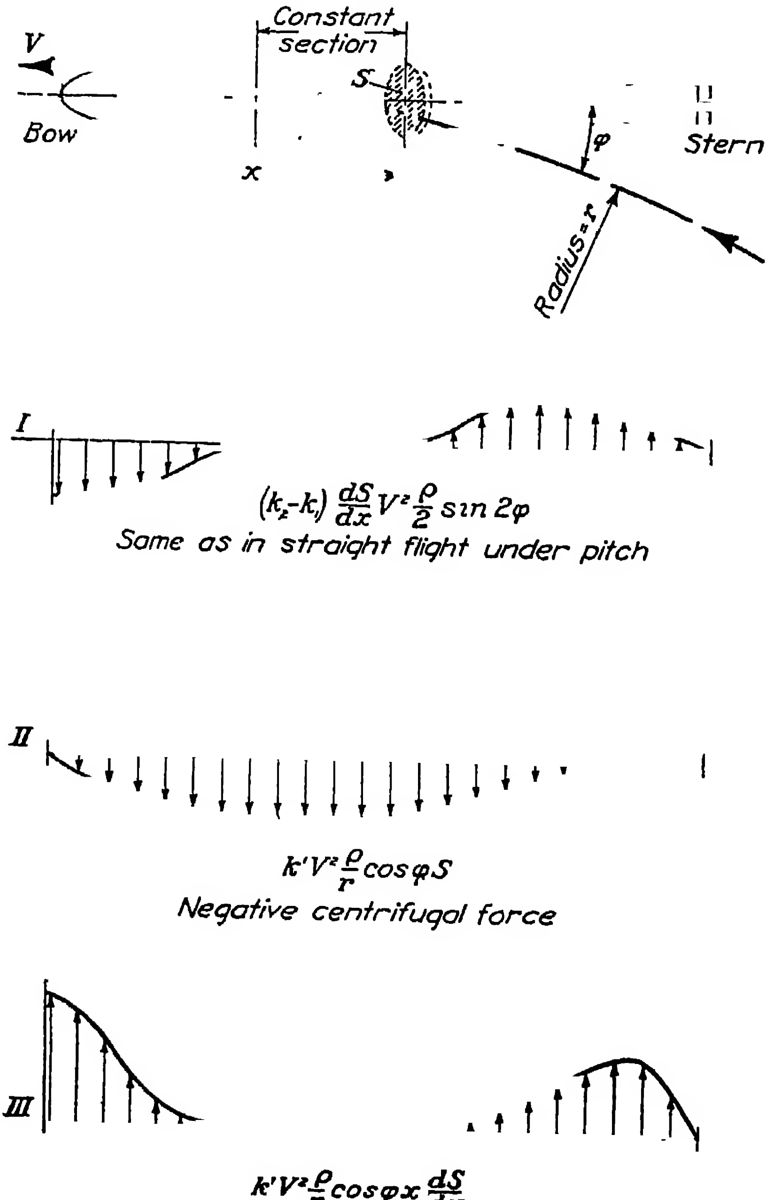

Figure 1 Diagram of the Transverse Airship Forces to be Added

The transverse component of the centrifugal force produced by the air taken along with the ship due to its longitudinal mass is neglected. Its magnitude is small; the distribution is discussed in author's papers and the discussion may be here omitted.[2]

The entire transverse force on an airship, turning under an angle of yaw with the velocity V and a radius r, is, according to the preceding discussion,

$$dF = dx\left[(k_2 - k_1)\frac{dS}{dx} \quad V^2\frac{\rho}{2} \quad \sin 2\varphi + k' V^2\frac{\rho}{r}S \quad \cos^2\varphi + k' V^2\frac{\rho}{r}x\frac{dS}{dx} \quad \cos^2\varphi\right] \qquad (5)$$

This expression of course does not contain the air forces on the fins.

15. Pressure Distribution Over the Bow of the Hull

Experiments have shown[2] that the pressure distribution near the bow of an airship hull, for all practical purposes agrees with the pressure distribution over the front of that ellipsoid, which approaches nearest the shape of the bow, and moves like it. This holds for all moderate angles of pitch or yaw, not only for motions parallel to the principal axis. The knowledge of the pressure distribution over a moving ellipsoid is, therefore, of definite value to the airship designer.

The mathematical analysis leading to the computation of this pressure distribution is somewhat involved, and its knowledge is not essential for the understanding of the following part of this book. Referring, therefore, to the existing textbooks on hydrodynamics for this analysis, for instance, to Lamb's Hydrodynamics (Fifth Edition, Section 103 and following), we will confine ourselves to state here a lemma, that is implicitly contained in equation 114 (8) in Lamb's treatise. This lemma commits itself easily to memory, and it is sufficient for the reduction of the following theorems and for the determination of the desired pressure distribution.

[2] See Author's Papers No. 19, page 177.

"If an ellipsoid is moving with uniform velocity parallel to one of its principal axes, say parallel to the x axis, the velocity potential at any point of the surface can be written in the form

$$\Phi = A'x \tag{6}$$

where A' is constant for a given flow and a given ellipsoid."

This theorem is the key to all the relations referring to the distribution of velocity and of pressure.

If the velocity of flow is not parallel to a principal axis, but has components in the direction of each of them, the resulting flow is the superposition of three flows analogous to the one just considered. Hence, at all points of the surface, the potential is a linear function of the Cartesian coordinates x, y, and z again, and can be written in the form

$$\Phi = A'x + B'y + C'z \tag{7}$$

where the coordinate axes are chosen to coincide with the axes of the ellipsoid. Hence, the curves of equal potential Φ are again situated on parallel planes.

Now, suppose first the ellipsoid to be at rest and the fluid moving relative to it, as in a wind tunnel, or an airship moored in a gale. The change from the ellipsoid moving through the fluid otherwise at rest to the fluid passing by the stationary ellipsoid does not affect the validity of equation (7) except giving the constants A', B', and C' other values, say A'', B'', and C''. In the latter case (the body at rest), furthermore, the velocity of the fluid at all points of the surface is parallel to the surface.

Consider first the elements of surface containing a line element at right angles to the planes of constant potential, i.e., at the points of the ellipsoid where the plane $A''x + B''y + C''z = 0$ meets the surface. It is, therefore, apparent from (7) that at all these points the velocity has the components A'', B'', and C''. This is evidently the maximum velocity.

At all other points of the ellipsoid the elements of surface are inclined towards the direction of maximum velocity, say by the angle ϵ. Then the elements of distance on the surface, s, between curves of equal potential are increased in the ratio $1/\cos\epsilon$ when compared with the actual distances between the planes of equal potential. Accordingly, the velocity, being equal to $\partial\Phi/\partial s$ is decreased inversely, its magnitude is $A''\cos\epsilon$. It will be noted in particular that the velocity is equal at surface elements which are inclined by the same angle ϵ. It is equal to the projection of the maximum velocity at right angles to the surface element. Hence the velocity cannot exceed the one rightly denoted by "maximum velocity," having the components A'', B'', and C''.

Returning now to the case when the direction of flow is parallel to a principal axis, we proceed to show that the maximum velocity A' stands in a very simple relation to the kinetic energy of the flow, and hence, to the apparent additional mass of the ellipsoid. We have now to suppose the fluid to be at rest and the ellipsoid to move, say, with the velocity U, parallel to a principal axis, e.g., the x axis. The kinetic energy of the flow set up is equal to

$$2\quad -\frac{\rho}{2}\int \Phi\frac{\partial\Phi}{\partial n}ds$$

i.e., the volume of fluid displaced by an element of the surface per unit of time, multiplied by the potential at the point of displacement and by $\rho/2$ where ρ denotes the density of the fluid. Now, the volume displaced by a surface element per unit time is equal to the projection of this element perpendicular to the direction of x, multiplied by the velocity U. The potential being $A'x$ the integrand becomes $A'Ux\times dydz$. $\int xdydz$ is the volume of the ellipsoid, hence the integral gives Volume $A'U\rho/2$. This is the kinetic energy, usually expressed by Volume $k_1U^2\rho/2$, where k_1 denotes the factor of apparent mass. It follows that:

$$A'/U = k_1$$

A'', referring to the case when the ellipsoid is stationary, is connected with A' by the equation

$$A'' = A' + U$$

as the latter flow results from the former by the superposition of the constant velocity U. Hence it appears that

$$A''/U = k_1 + 1 = A$$

A is the maximum velocity corresponding to a flow having unit velocity along the x axis. It is a constant for a given ellipsoid.

A equals the sum of 1 and of the factor of apparent mass k_1. This is confirmed for two special cases, where the factor A is well known. With a sphere, the maximum velocity is 1.5 times the velocity of flow, and the additional apparent mass is one-half the mass of the displaced fluid. With a circular cylinder, moving at right angles to its axis, the maximum velocity is twice the velocity of flow, and the apparent additional mass is equal to the mass of the displaced fluid.

The ratios, $A = A''/U$, $B = B''/V$, $C = C''/W$, are independent of the velocity components U, V, and W and hence only depend on the ratio of the three semi-axes of the ellipsoid, a, b, and c. Lamb gives the method to compute them. Compute first the integral

$$\alpha = abc \int_0^\infty \frac{dx}{(a^2 + x)\sqrt{(a^2 + x)(b^2 + x)(c^2 + x)}}.$$

and the analogous integrals β and γ for the axes b and c. The factors of apparent mass are then

$$k_1 = \alpha/(2 - \alpha), \text{ etc.}$$

There are no tables for A, B, C, or for k_1, k_2, k_3 published yet. The integrals for a, etc., can be numerically evaluated, however, in each single case, hence the absence of these tables does not interfere with the practical application of the method. For the

special case $b = c$, that is, for ellipsoids of revolution, k_1 and k_2 have been computed for a series of elongation ratios, and are reprinted in Table I. They are connected with each other by the relation

$$k_1 = \frac{1 - k_2}{2k_2}$$

The determination of the velocity at any point is thus reduced to a simple geometric problem. The maximum velocity, whose components are AU, BV, and CW, has to be projected onto the plane tangent to the ellipsoid at the point considered, i.e., it must be multiplied by the cosine of the angle between the normals to the surface at this point and at the point where the velocity is a maximum.

TABLE I. COEFFICIENTS OF ADDITIONAL MASS OF ELLIPSOID

Length (diameters)	k_1 (longitudinal)	k_2 (transverse)	$k_2 - k_1$	k' (rotation)
1 00	0 500	0.500	0 000	0 000
1 50	.305	.621	.316	094
2 00	.209	.702	.493	.240
2 51	.156	.763	.607	.367
2 99	.122	803	.681	.465
3 99	.082	860	.778	.608
4 99	.059	.895	.836	.701
6 01	045	.918	.873	.764
6.97	.036	.933	.897	.805
8.01	.029	.945	.916	.840
9.02	.024	.954	.930	.865
9 97	.021	.960	.939	.883
∞	.000	1.000	1.000	1.000

In the most interesting case of an ellipsoid of revolution, this can be done analytically in a convenient way. The following formula is most easily obtained by the application of elementary vector analysis. First compute the component of the maximum velocity in a direction normal to the surface at a given point. Remember further, that the longitudinal component of the max-

imum velocity is AU, and the lateral component of the maximal velocity is BV.

Let now the angle between the normal and the longitudinal axis be δ and let the angle between the two planes passing through the longitudinal axis, the one passing through the point in question and the other being parallel to the direction of motion, be β. Then $\cos\delta$ is the longitudinal component and $\sin\delta\cos\beta$ the lateral component of the normal of unit length. Hence, the component of the maximum velocity in a direction perpendicular to the element of surface is

$$V_2 = (1+k_1)U\cos\delta + (1+k_2)V\sin\delta\cos\beta$$

Let V_1 denote the component parallel to the surface element. Then

$$V_1^2 + V_2^2 = V_{max}^2 \text{ and hence}$$
$$V_1 = \sqrt{V_{max}^2 - V_2^2}$$
$$V_1 =$$
$$\sqrt{(1+k_1)^2U^2 + (1+k_2)^2V^2 - [(1+k_1)U\cos\delta + (1+k_2)V\sin\delta\cos\beta]^2} \quad (8)$$

This is the desired general formula for the velocity of flow along the surface. The pressure is then computed directly from this velocity at the points of the ellipsoid, now supposed to be stationary in the flowing fluid. For this is a steady flow, and hence Bernouilli's equation for the pressure holds true, viz.: $p + (\frac{1}{2}\rho V_1^2) = \text{const.}$ That is, the pressure is equal to an arbitrary constant pressure minus $V_1^2\rho/2$, where V_1 denotes the velocity. The points of greatest velocity are those of smallest pressure or of greatest suction. The curves of equal velocity are also the curves of equal pressure.

In practice, we are chiefly interested in rather elongated ellipsoids of revolution and the angle α between the principal axis and the direction of motion is small. With very elongated ellipsoids, k_2 is about 1 and k_1 is very small. Hence B is about 2 and A is about 1. The components of the maximum velocity become U and $2V$. Comparing them with the components of

the motion U and V, it follows that the angle between the direction of the maximum velocity and the axis is about twice as large as the angle between the direction of motion and the axis.

16. Distribution of Pressure Around the Cross-Section of the Hull

The pressure around circular cross-sections far enough from the ends of an elongated hull is distributed as the sine of the angle; that is, proportional to the distance from a diameter. This knowledge gives an estimate about the pressure differences to be expected, and is useful for the interpretation of pressure distribution measurements.

The relation is based on the lemma of the preceding section. This lemma holds for elliptical cylinders, and thus for circular cylinders as special cases. The two-dimensional velocity potential of the flow created by the rectilinear motion assumes values at the circumference proportional to the distance from a plane parallel to the axis. If the potential is chosen zero at the center, the potential at the circumference is proportional to the distance from a plane through the axis. With a circle, this is the plane at right angles to the motion.

This simple relation is applied to the pressure distribution around the cross-section of an airship hull as follows. Let U again denote the velocity of flight, u the longitudinal component, and v and w the two lateral components of the air motion caused by the motion of the hull. Hence the square of the air velocity relative to the hull is $(U + u)^2 + v^2 + w^2$. u, v, and w are small when compared with U, the hull being assumed to be elongated. The square contains as the largest term U^2, but this is constant and does not give rise to pressure differences. The only term of next order is then $2Uu$; the remaining terms are small when compared with this term. The square of the velocity, just analyzed, is proportional to the pressure according to Bernouilli's theorem. It appears, then, that the pressure measured from a

suitable standard is proportional to the component u, of the air velocity caused by the motion of the hull.

This longitudinal velocity component, now, varies around the cross-section as the potential does, provided all cross-sections are geometrically similar. We imagine the potential to be computed from the two-dimensional flow around the cross-section. If all cross-sections are similar, for instance circular, the potential can be written as the product of two factors, the one constant on meridians and the other on each cross-section. But if the potential distributions on different cross-sections differ by a constant factor only their differences will do the same, and it follows that at each cross-section u is proportional to the potential. Hence, with elongated hulls having similar cross-sections, the pressure is proportional to the potential and that proves the relation.

17. Size of the Fins

The model tests seem to indicate that the actual unstable moment of the hull in air agrees nearly with that in a perfect fluid. Now the actual airships with fins are statically unstable (as the word is generally understood, not aerostatically of course), but not much so, and for the present general discussion it can be assumed that the unstable moment of the hull is nearly neutralized by the transverse force of the fins. We saw this unstable moment to be $M =$ (Volume) $(k_2 - k_1)$ $(V^2\rho/2)$ sin 2φ, where $(k_2 - k_1)$ denotes the factor of correction due to finite elongation. Its magnitude is discussed in the first part of this chapter. Hence the transverse force of the fins must be about M/a where a denotes the distance between the fin and the center of gravity of the ship. Then the effective area S of the fins—that is, the area of a wing giving the same lift in a two-dimensional flow—follows:

$$\frac{(\text{Volume})(k_2 - k_1)}{a\pi}$$

Taking into account the span b of the fins—that is, the distance of two utmost points of a pair of fins—the effective fin area S must be[3]

$$\frac{(\text{Volume})(k_2 - k_1)}{a} \times \frac{1 + 2S/b^2}{\pi}$$

This area S, however, is greater than the actual fin area. Its exact size is uncertain, but a far better approximation than the fin area is obtained by taking the projection of the fins and the part of the hull between them. This is particularly true if the diameter of the hull between the fins is small.

If the ends of two airships are similar, it follows that the fin area must be proportional to $(k_2 - k_1)$ (Volume)$/a$. For rather elongated airships $(k_2 - k_1)$ is almost equal to 1 and constant, and for such ships, therefore, it follows that the fin area must be proportional to (Volume)$/a$, or, less exactly, to the greatest cross-section, rather than to (Volume)$^{2/3}$. Comparatively short ships, however, have a factor $(k_2 - k_1)$ rather variable, and with them the fin area is more nearly proportional to (Volume)$^{2/3}$.

This refers to circular section airships. Hulls with elliptical section require greater fins parallel to the greater plan view. If the greater axis of the ellipse is horizontal, such ships are subjected to the same bending moments for equal lift and size, but the section modulus is smaller, and hence, the stresses are increased. They require, however, a smaller angle of attack for the same lift. The reverse holds true for elliptical sections with the greater axes vertical.

18. Flying in a Circle

If the airship flies along a circular path, the centrifugal force must be neutralized by the transverse force of the fin, for only the fin gives a considerable resultant transverse force. At the same time, the fin is supposed nearly to neutralize the unstable

[3] *Cf.* Chapter V, Section 39.

moment. We saw now that the angular velocity, though indeed producing a considerable change of the distribution of the transverse forces, and hence of the bending moments, does not give rise to a resulting force or moment. Hence, the ship flying along the circular path must be inclined by the same angle of yaw as if the transverse force is produced during a rectilinear flight by pitching. From the equation of the transverse force

$$\text{Vol.}\rho\frac{V^2}{r} = \frac{\text{Vol}\,(k_2 - k_1)(V^2\,\rho/2)\,\sin 2\varphi}{a}$$

it follows that the angle is approximately

$$\Phi \sim \frac{a}{r}\,\frac{1}{k_2 - k_1}$$

This expression in turn can be used for the determination of the distribution of the transverse forces due to the inclination. The resultant transverse force is produced by the inclination of the fins. The rotation of the rudder has chiefly the purpose of neutralizing the damping moment of the fins themselves.

From the last relation, substituted in equation (5), follows approximately the distribution of the transverse forces due to the inclination of pitch, consisting of two parts. The first part is

$$\frac{dS}{dx}V^2\frac{\rho l a}{2r}dx \tag{9}$$

The other part is due to the angular velocity; it is approximately

$$k'\frac{2x\,dS}{r\,dx}V^2\frac{\rho}{2}dx + k'\frac{V^2\rho}{r}S\,dx \tag{10}$$

The first term in formula (10) together with formula (9) gives a part of the bending moment. The second term in (10) having mainly a direction opposite to the first one and to the centrifugal force, is almost neutralized by the centrifugal forces of the ship and gives additional bending moments not very considerable either. It appears, then, that the ship experiences smaller

bending moments when creating an air force by yaw opposite to the centrifugal force than when creating the same transverse force during a straight flight by pitch. For ships with elliptical sections this cannot be said so generally. The second term in (10) will then less perfectly neutralize the centrifugal force, if that can be said at all, and the bending moments become greater in most cases.

19. Flying Through Gusts

Most airship pilots are of the opinion that severe aerodynamic forces act on airships flying in bumpy weather. An exact computation of the magnitude of these forces is not possible, as they depend on the strength and shape of the gusts and probably no two exactly equal gusts ever occur. Nevertheless, it is worth while to reflect on this phenomenon and to get acquainted with the underlying general mechanical principles. It will be possible to determine how the magnitude of the velocity of flight influences the air forces due to gusts. It even becomes possible to estimate the magnitude of the air forces to be expected, though this estimation will necessarily be somewhat vague, due to ignorance about the gusts.

The airship is supposed to fly not through still air but through an atmosphere the different portions of which have velocities relative to each other. This is the cause of the air forces in bumpy weather, the airship coming in contact with portions of air having different velocities. Hence, the configuration of the air flow around each portion of the airship is changing as it always must to conform to the changing relative velocity between the portion of the airship and the surrounding air. A change of the air forces produced is the consequence.

Even an airship at rest experiences aerodynamical forces in bumpy weather, as the air moves toward it. This is very pronounced near the ground, where the shape of the surrounding objects gives rise to violent local motions of the air. The pilots have the impression that at greater altitudes an airship at rest

does not experience noticeable air forces in bumpy weather. This is plausible. The hull is struck by portions of air with relatively small velocity, and as the forces vary as the square of the velocity they cannot become large.

It will readily be seen that the moving airship cannot experience considerable air forces if the disturbing air velocity is in the direction of flight. Only a comparatively small portion of the air can move with a horizontal velocity relative to the surrounding air and this velocity can only be small. The effect can only be an air force parallel to the axis of the ship which is not likely to create large structural stresses.

There remains, then, as the main problem, the airship in motion coming in contact with air moving in a transverse direction relative to the air surrounding it a moment before. The stresses produced are more severe if a larger portion of air moves with that relative velocity. Therefore, it is logical to consider portions of air that are large compared with the diameter of the airship; smaller gusts produce smaller air forces. It is now essential to realize that their effect is exactly the same as if the angle of attack of a portion of the airship is changed. The air force acting on each portion of the airship depends on the relative velocity between this portion and the surrounding air. A relative transverse velocity u means an effective angle of attack of that portion equal to u/V, where V denotes the velocity of flight. The airship, therefore, is now to be considered as having a variable effective angle of attack along its axis. The magnitude of the superposed angle of attack is u/V, where u generally is variable. The momentum produced at each portion of the airship is the same as the air force at that portion if the entire airship would have that particular angle of attack.

The magnitude of the air force depends on the conicity of the airship portion as described in Section 2. The force is proportional to the angle of attack and to the square of the velocity of flight. In this case, however, the superposed part of the angle of attack varies inversely as the velocity of flight. It

results, then, that the air forces created by gusts are directly proportional to the velocity of flight. Indeed, as we have shown, they are proportional to the product of the velocity of flight and the transverse velocity relative to the surrounding air.

A special and simple case to consider for a closer investigation is the problem of an airship flying from air at rest into air with constant transverse horizontal or vertical velocity. The portion of the ship already immersed has an angle of attack increased by the constant amount u/V. Either it can be assumed that by operation of the controls the airship keeps its course or, better, the motion of an airship with fixed controls and the air forces acting on it under these conditions can be investigated. As the fins come under the influence of the increased transverse velocity later than the other parts, the airship is, as it were, unstable during the time of entering into the air of greater transverse velocity and the motion of the airship aggravates the stresses.

In spite of this the actual stresses will be of the same range of magnitude as if the airship flies under an angle of pitch of the magnitude u/V, for in general the change from smaller to greater transverse velocity will not be so sudden and complete as supposed in the last paragraph. It is necessary chiefly to investigate the case of a vertical transverse relative velocity u. The most severe condition for the airship is a considerable angle of pitch, and a vertical velocity u increases these stresses. Hence, it would be extremely important to know the maximum value of this vertical velocity. The velocity in question is not the greatest vertical velocity of portions of the atmosphere occurring, but differences of this velocity within distances smaller than the length of the airship. It is very difficult to make a positive statement as to this velocity, but it is necessary to conceive an idea of its magnitude, subject to a correction after the question is studied more closely. Studying the meteorological papers in the reports of the British Advisory Committee

for Aeronautics, chiefly those of 1909–10 and 1912–13, I should venture to consider a sudden change of the vertical velocity by 2 m./sec. (6.5 ft./sec.) as coming near to what to expect in very bumpy weather. The maximum dynamic lift of an airship is produced at low velocity, and is the same as if produced at high velocity at a comparatively low angle of attack, not more than 5°. If the highest velocity is 30 m./sec. (67 mi./hr.), the angle of attack u/V, repeatedly mentioned before, would be

$$57.3 \times 2/30 = 3.8^\circ$$

This is a little smaller than 5°, but the assumption for u is rather vague. It can only be said that the stresses due to gusts are of the same range of magnitude as the stresses due to pitch, and they may exceed them.

A method for keeping the stresses down in bumpy weather is by slowing down the speed of the airship. This is a practice common among experienced airship pilots. This procedure is particularly recommended if the airship is developing large dynamic lift, positive or negative, as then the stresses are already large.

20. Synopsis

The resultant moment of the airship hull flying with the angle of attack α and the velocity V is

$$M = (K_2 - K_1) \quad \sin 2\alpha \quad V^2\rho/2$$

In a circular path, the angle of attack must be measured at the center of volume.

This resultant moment is caused by forces along the axis distributed according to equation (5). The last two terms in the bracket of this equation vanish at straight flight.

The pressure around an elliptic cross-section is a linear function of the distance from a diameter. For circular sections, this diameter is at right angles to the plane of pitch.

The pressure distribution over the bow is more complicated.

For an elliptic bow, the velocity distribution is given by the lengthy formula (8) theorem. All bows can be approximated by elliptic ones. For very elongated bows with circular cross-sections divide the velocity of motion into components u and v parallel and at right angles to the axis. The maximum velocity has then the components u and $2v$. The lowest pressure occurs at a point where the tangential plane is parallel to this maximum velocity.

The fin area of very elongated airships is proportional to the cross-section and independent of the length of the hull. With less elongated hulls the fin area becomes smaller than this rule would indicate.

The stresses caused by gusts are larger than those caused by maneuvering.

21. Problems and Suggestions

An airship hull has the shape of an ellipsoid of revolution, the largest diameter is 70 ft. and the length is 700 ft. It is flying at a velocity of 90 ft./sec. in air of a density of 1/420 lb. sec²/ft⁴.

1. With elongated ellipsoids flying in pitch, the transverse air force per unit length is proportional to the distance from the center.

2. How large is the resultant moment of the hull if the ship is flying at an angle of pitch of 4°?

3. How large is the transverse air force between the cross-sections 200 ft. and 300 ft. from the bow?

4. How large is the difference of pressure at the highest and lowest point of the cross-section 250 ft. from the bow?

5. The resultant of the lift created by the fins intersects with the axis 625 ft. from the bow. How large is the lift of the airship in Problem 2?

6. How much is the largest pressure on the hull in Problem 2?

7. Compute the lowest pressure, using the method for infinite elongation.

8. Compute the same using the exact method and Table I.

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7. Compute the lowest pressure, using the method for infinite elongation.

8. Compute the same using the exact method and Table I.

9. Where are the points of the lowest and highest pressures at the bow?

10 The airship flies without pitch. How large is the kinetic energy of the potential flow?

11. The ship is turning without pitch with a radius of turn of 1,000 ft. How large is the resultant moment?

12. How large is the force caused by the displacement of the rudders?

13. The ıateral air force load on a conical portion of a hull flying in pitch is a linear function of the distance from the bow.

CHAPTER III

THE POTENTIAL FLOW OF THE STRAIGHT LINE

22. Outlook

We proceed to the study of the two-dimensional flow of the straight line. This is not only the flow caused by the motion of a rigid straight line, but in addition all the infinitely many flows determined by an arbitrary distribution of the normal velocity of the potential itself along the points of a straight line. Such flows represent the motion of a perfect fluid caused by a line that is only momentarily straight and in the process of distorting itself.

This comparatively simple problem requires additional mathematical tools, the complex numbers. They are here indispensable, and on the other hand, suggest themselves very naturally to this application.

Each two-dimensional solution of Laplace's equation possesses a complementary solution which multiplied by the imaginary unit i and added to the primary solution forms an analytical function of the complex variable representing the coordinates of the plane. Inversely: each such function, merely by being a function of the complex coordinates, consists of two solutions of Laplace's equation. Its real part is a potential of a two-dimensional potential flow of an incompressible fluid. This is the desired special solution if it complies with the boundary conditions.

The complex functions representing potential flows of the straight line are simple, much simpler than the potentials themselves. We multiply them by constants and add them in a series in order to obtain more general solutions complying with any boundary conditions along the straight line. This series

is found to be a Fourier series, hence it is easy to determine its constants. The series is at last transformed into a definite integral for the numerical applications.

23. Potential Functions

We show in the next chapters how the potential flows applied to the solution of airplane problems, although three-dimensional by themselves, can be built up from two-dimensional flows. We shall immediately show how these two-dimensional potentials can be represented by a function of one variable only. This brings the simplification of the problem to an extreme. We started from a three-dimensional velocity distribution, that is, from three velocity components each of them being a function of three space coordinates. We reduced first the number of the dependent variables from three to one, by the introduction of the potential. We reduce then the number of the independent variables from three to two, building up the three-dimensional flow from two-dimensional ones. We at last reduce the number of the independent variables also to one by the introduction of the complex numbers, and shall then have reduced three functions of three variables to one function of one variable.

The advantage of having to do with one function of one variable only is so great, and moreover this function in practical cases becomes so much simpler than any of the functions which it represents, that it would pay to get acquainted with this method even if the reader had never occupied himself with complex numbers before. The matter is simple and can be explained in a few words.

The ordinary or real numbers, x, are considered to be the special case of more general expressions $(x + iy)$ in which y happens to be zero. If y is not zero, such an expression is called a complex number. x is its real part, iy is its imaginary part and consists of the product of y, any real or ordinary number by the quantity i, which latter is the solution of

$$i^2 = -1;\text{ i.e., } i = \sqrt{-1}$$

The complex number $x + iy$ can be supposed to represent the point of the plane with the coordinates x and y, and that may be in this paragraph the interpretation of a complex number. So far, the system would be a sort of vector symbolism, which indeed it is. The real part x is the component of a vector in the direction of the real axis, and the factor y of the imaginary part iy is the component of the vector in the y direction. The complex numbers differ, however, from vector analysis by the peculiar fact that it is not necessary to learn any new sort of algebra or calculus for this vector system. On the contrary, all rules of calculation, valid for ordinary numbers, are also valid for complex numbers without any change whatsoever.

The addition of two or more complex numbers is accomplished by adding the real parts and imaginary parts separately.

$$(x + iy) + (x' + iy') = (x + x') + i(y + y')$$

This amounts to the same process as the superposition of two forces or other vectors. The multiplication is accomplished by multiplying each part of the one factor by each part of the other factor and adding the products obtained. The product of two real factors is real, of course. The product of one real factor and one imaginary factor is imaginary, as appears plausible. The product of $i \times i$ is taken as -1, and hence the product of two imaginary parts is real again. Hence the product of two complex numbers is in general a complex number again

$$(x + iy)(x' + iy') = (xx' - yy') + i(xy' + x'y)$$

There is now one trick, as we may say, which the student must know in order to get the advantage of the use of complex numbers. That is the introduction of polar coordinates. The distance of the point (x, y) from the origin $(0, 0)$ is called R and the angle between the positive real axis and the radius vector from the origin to the point is called φ, so that $x = R \cos \varphi$, $y = R \sin \varphi$. Multiply now

$$(R_1 \cos\varphi_1 + iR_1 \sin\varphi_1)(R_2 \cos\varphi_2 + iR_2 \sin\varphi_2)$$

The result is

$$R_1R_2 \cos\varphi_1 \cos\varphi_2 - R_1R_2 \sin\varphi_1 \sin\varphi_2 + i(R_1R_2 \cos\varphi_1 \sin\varphi_2 + R_1R_2 \sin\varphi_1 \cos\varphi_2)$$

or, otherwise written

$$R_1R_2[\cos(\varphi_1 + \varphi_2) + i \sin(\varphi_1 + \varphi_2)]$$

That is: The radius R of the product is the product of the radii R_1 and R_2 of the two factors; the angle φ of the product is the sum of the angles φ_1 and φ_2 of the two factors, from which follows:

$$(\cos\varphi + i \sin\varphi)^n = \cos n\varphi + i \sin n\varphi$$

This is Moivre's formula.

We proceed now to explain how these complex numbers can be used for the representation of a two-dimensional potential flow. It follows from the very fact that a function of the complex number, which in general is also a complex number, can be treated exactly like the ordinary real function of one real variable, given by the same mathematical expression. In particular it can be differentiated at each point and has then one definite differential quotient, the same as the ordinary function of one variable of the same form. The process of differentiation of a complex function is indeterminate, in so far as the independent variable $(x + iy)$ can be increased by an element $(dx + idy)$ in very different ways, viz., in different directions. The differential quotient is, as ordinarily, the quotient of the increase of the function divided by the increase of the independent variable. One can speak of a differential quotient at each point only if the value results the same in whatever direction of $(dx + idy)$ the differential quotient is obtained. It must be the same, in particular, when dx or dy is zero.

The function to be differentiated may be

$$F(x + iy) = G(x,y) + iH(x,y)$$

where both G and H are real functions of x and y. The differentiation gives

$$\partial F/\partial x = \partial G/\partial x + i\, \partial H/\partial x$$

or again

$$\partial F/i\partial y = -i\,\partial G/\partial y + \partial H/\partial y$$

These two expressions must give identical results and hence must be equal. That is, both the real parts and both the imaginary parts are equal:

$$\partial G/\partial x = \partial H/\partial y; \quad -\partial G/\partial y = \partial H/\partial x$$

Differentiating these equations with respect to dx and dy;

$$\partial^2 G/\partial x\partial y = \partial^2 H/\partial y^2 = -\partial^2 H/\partial x^2; \text{ i.e., } \partial^2 H/\partial x^2 + \partial^2 H/\partial y^2 = 0$$

or again

$$\partial^2 H/\partial x\partial y = \partial^2 G/\partial y^2 = -\partial^2 G/\partial x^2; \text{ i.e., } \partial^2 G/\partial x^2 + \partial^2 G/\partial y^2 = 0$$

Hence, it appears that the real part as well as the imaginary part of any analytical complex function complies with Laplace's equation for the potential of an aerodynamic flow, and hence, can be such a potential. If the real part is this potential, we will call the complex function the "potential function" of the flow. It is not practical, however, to split the potential function in order to find the potential and to compute the velocity from the potential. The advantage of having only one variable would then be lost. It is not the potential that is used for the computation of the velocity, but instead of it the potential function directly. It is easy to find the velocity directly from the potential function. Differentiate $F(x + iy) = F(z)$. It is seen that when $dz = dx$,

$$dF(z)/dz = \partial G/\partial x + i\,\partial H/\partial x$$

But it was shown before that

$$\partial H/\partial x = -\partial G/\partial y$$

Hence,

$$dF(z)/dz = \partial G/\partial x - i\,\partial G/\partial y$$

The velocity has the components $\partial G/\partial x$ and $\partial G/\partial y$, being the gradient of the potential G, by definition. Written as a complex vector, it would be

$$\partial G/\partial x + i\,\partial G/\partial y$$

It appears, therefore:

Any analytical function $F(z)$ can be used for the representation of a potential flow. The potential of this flow is the real part of this potential function, and its differential quotient dF/dz called the "velocity function," represents the velocity at each point "turned upside down." That means that the component of the velocity in the direction of the real axis is given directly by the real part of the velocity function dF/dz and the component of the velocity at right angles to the real axis is equal to the reversed imaginary part of dF/dz. The absolute magnitude of the velocity is equal to the absolute magnitude of dF/dz.

24. Flows Around a Straight Line

We proceed now to the series of two-dimensional flows which are of chief importance for the solution of the aerodynamic

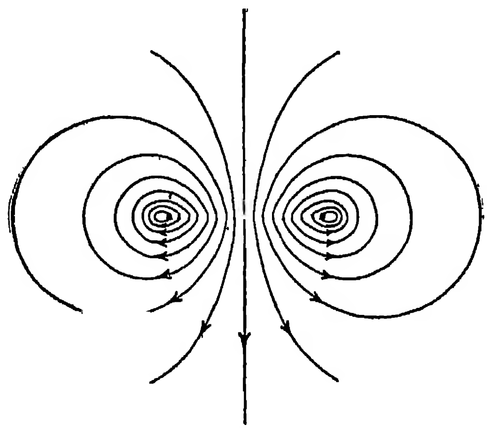


Figure 2. Transverse Flow of the Straight Line

problems in practice. They stand in relation to the straight line. The privileged position of the straight line rests on the fact that both the front view and the cross-section of a wing are approximately described by a straight line. The different

types of flow to be discussed in this section have in common that at the two ends of a straight line, but nowhere else, the velocity may become infinite. At infinity it is zero. This suggests the potential function $\sqrt{z^2 - 1}$, which has discontinuities at the points ± 1 only; but it does not give the velocity zero at infinity.

$$F = \sqrt{z^2 - 1} - z$$

gives rise to an infinite velocity at the points $z = \pm 1$, which may be regarded as the ends of the straight line, and in addition the velocity becomes zero at infinity. A closer examination shows that indeed the potential function

$$F = i(z - \sqrt{z^2 - 1}) \tag{1}$$

represents the flow produced by the straight line extending between the points $z = \pm 1$, moving transversely in the direction of the negative imaginary axis with the velocity 1 in the fluid otherwise at rest. For, its velocity function is

$$F' = i - \frac{zi}{\sqrt{z^2 - 1}}$$

giving for points on the line a transverse velocity 1. This flow may be called "transverse flow." The velocity potential at the points of the line, i.e., for $y = 0$ is $\sqrt{1 - x^2}$. This gives the kinetic energy of the flow (half the integral of the product of potential, density and normal velocity component, taken around the line):

$$T = \frac{\rho}{2} \cdot 2 \int_{-1}^{+1} \sqrt{1 - x^2}\, dx = \frac{\rho}{2}\pi \tag{2}$$

giving an apparent mass of the straight line moving transversely equal to the mass of the fluid displaced by a circle over the straight line as diameter.

The flow around the straight line just discussed can be considered as a special case of a series of more general flows, represented by the potential function

$$F = i(z - \sqrt{z^2 - 1})^n \tag{3}$$

where n is any positive integer. $n = 1$ gives the transverse flow considered before. For n different from 1 the component of the transverse velocity along the straight line is no longer constant, but variable and given by a simple law. Such a flow, therefore, cannot be produced by a rigid straight line moving, but by a flexible line, being initially straight and in the process of distorting itself.

It is helpful to introduce as an auxiliary variable the angle δ defined by $z = \cos\delta$. Then the potential function is

$$F = i(\cos n\delta - i \sin n\delta)$$

where δ is, of course, complex. The potential along the line is

$$\Phi = \sin n\delta \tag{4}$$

where now δ is real. The velocity function is

$$F' = dF/dz = dF/d\delta \cdot d\delta/dz = -n/\sin\delta \cdot (\cos n\delta - i \sin n\delta)$$

giving at points along the line the transverse component

$$u = -n \sin n\delta/\sin\delta \tag{5}$$

and the longitudinal component

$$v = -n \cos n\delta/\sin\delta \tag{6}$$

This becomes infinite at the two ends. The kinetic energy of the flow is

$$T = \tfrac{1}{2} n\rho \int_0^{2\pi} \frac{\sin^2 n\delta}{\sin\delta} \sin\delta\, d\delta = n\pi\frac{\rho}{2} \tag{7}$$

The momentum is given by the integral

$$\rho \int \Phi dx$$

to be taken along both sides of the straight line, since the velocity potential times ρ represents the impulsive pressure necessary to create the flow. This integral becomes

$$\rho \int_0^{2\pi} \sin n\delta \sin\delta d\delta$$

This is zero except for $n = 1$.

By the superposition of several or infinitely many of the flows of the series discussed

$$F = i[A_1(z - \sqrt{z^2 - 1}) + A_2(z - \sqrt{z^2 - 1})^2 + \ . \ . \quad + A_n(z - \sqrt{z^2 - 1})^n] \quad (8)$$

with arbitrary intensity, infinitely many more complicated flows around the straight line can be described. There is even no potential flow of the described kind around the straight line existing which cannot be obtained by such superposition. The kinetic energy of the flow obtained by superposition stands in a very simple relation to the kinetic energy of the single flows, which relation is by no means self-evident. It is the sum of them. This follows from the computation of the kinetic energy by integrating the product of the transverse component of velocity and the potential along the line. This kinetic energy is

$$T = \frac{\rho}{2}\int_0^{2\pi} (A_1 \sin\delta + A_2 \sin 2\delta + \ . \ . \ .) (A_1 \sin\delta + 2A_2 \sin 2\delta + \ . \ . \ .)d\delta \quad (9)$$

But the integral

$$\int_0^{\pi} \sin n\delta \sin m\delta d\delta = 0 \quad (n \neq m)$$

is zero if m and n are different integers. For, integrating two times by parts gives the same integral again, multiplied by $(m/n)^2$. In the same way it can be proved that

$$\int_0^{\pi} \cos m\delta \cos n\delta d\delta = 0 \quad (m \neq n)$$

Only the squares in integral (9) contribute to the energy and each of them gives just the kinetic energy of its single term (equation 7).

It may happen that the distribution of the potential Φ along the line is given, and the flow determined by this distribution is to be expressed as the sum of flows (equation 8). The condition is: At points on the line, a known function of x is given, and is to be expanded into

$$\Phi = A_1 \sin\delta + A_2 \sin 2\delta + \ldots + A_n \sin n\delta + \ldots \tag{10}$$

The coefficients A are to be determined. The right-hand side of equation (10) is called a Fourier's series, and it is proved in the textbooks that the coefficients A can always be determined as to conform to the condition, if Φ has reasonable values. At the ends $\delta = 0$ or π, hence Φ has to be zero there as then all sines are zero.

Otherwise expressed, equation (3) gives enough different types of flow to approximate by means of superposition any reasonable distribution of the potential over a line, with any exactness desired. This being understood, it is easy to show how the coefficients A can be found.

Integrate

$$\int_0^\pi (A_1 \sin\delta + A_2 \sin 2\delta \ldots) \sin n\delta d\delta$$

According to equation (11) all integrals become zero with the exception of

$$A_n \int_0^\pi \sin^2 n\delta d\delta = \frac{\pi}{2} A_n$$

Hence

$$A_n = \frac{2}{\pi}\int_0^\pi \Phi \sin n\delta d\delta \tag{11}$$

These values may be introduced into equation (8), and thus the potential function F is determined.

Another problem of even greater practical importance is to determine the potential functions, equation (3), which superposed give a desired distribution of the transverse component of velocity. The condition now is: u is given and is to be expanded as follows:

$$u = A_1 \frac{\sin\delta}{\sin\delta} + 2A_2 \frac{\sin 2\delta}{\sin\delta} + \ldots \tag{12}$$

which means that $u \sin \delta$, a known function, is to be expanded into a Fourier's series

$$u \sin\delta = B_1 \sin\delta + B_2 \sin 2\delta + \ldots + B_n \sin n\delta + \ldots \tag{13}$$

The B's may be determined by an equation like (11), and then the A's may be deduced, since

$$A_n = B_n/n \tag{14}$$

This is always possible if the velocity component is finite along the line. These values may then be introduced in equation (8).

The value of the potential function as given by series (10) with the values of A_n substituted from (14) may be transformed into a definite integral which sometimes is more convenient for application. Let u_0 be a function of the coordinate z_0, a point on the line joining $z = -1$ and $z = +1$, and let $f(z, z_0)$ be a function to be determined so that

$$F = \int_{-1}^{+1} f(z, z_0) u_0 dz_0$$

We find that this relation is satisfied by the function

$$f = \frac{1}{\pi}[\log (e^{i\delta} - e^{i\delta_0}) - \log (e^{i\delta_0} - e^{-i\delta_0})]$$

Hence, the velocity function is

$$F' = \frac{dF}{dz} = \int_{-1}^{+1} \frac{df}{dz} u_0 dz_0$$

That means superposition of infinitely many "elementary flows." An element (at z_0) of the line gives rise to a potential function $f(z,z_0)\, u_0 dz_0$ and to the velocity function $df/dz \cdot u_0 dz_0$,

$$f' = \frac{df}{dz} = \frac{1}{\pi} \frac{\sin\delta_0}{\sin\delta} \frac{1}{\cos\delta - \cos\delta_0} = \pm \frac{1}{\pi} \frac{1}{z - z_0} \sqrt{\frac{1 - z_0^2}{1 - z^2}}$$

where the plus sign is to be taken for points on the positive side of the line, and the negative sign for those on the opposite side. In this elementary flow, then, the velocity is parallel to the line at all points of the line excepting the point z_0, being directed away from this point on the positive side of the line and toward it on the other side. For points close to z_0,

$$f' u_0 dz_0 = \frac{u_0 dz_0}{\pi} \frac{1}{z - z_0}$$

from which the value of the velocity of the flow near z_0 may be deduced. If a small circle is drawn around the point z_0, it is

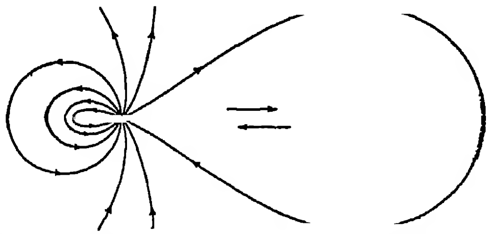


Figure 3. Flow Contributed by One Wing Section Element

seen that there is a flow out from the point z_0 of amount $u_0 dz_0$ per second on the positive side and an inflow of an equal amount on the other side; so that this is equivalent to there being a transverse velocity u_0 at points along the element dz_0, positive on one side, negative on the other. The total elementary flow around the line is illustrated in Figure 3.

Substituting the value of f' in F'; that is, superposing the elementary flows for all points z_0 of the line gives the velocity function

$$F' = \pm \int_{-1}^{+1} \frac{1}{\pi} \frac{u_0 dz_0}{z - z_0} \sqrt{\frac{1 - z_0^2}{1 - z^2}}$$

Therefore, for any point on the real axis, the transverse velocity is u_0 as is required and the longitudinal velocity

$$v_z = \pm \int_{-1}^{+1} \frac{1}{\pi} \frac{u_0 dz_0}{z - z_0} \sqrt{\frac{1 - z_0^2}{1 - z^2}} \tag{15}$$

The infinite velocity near the edge requires special attention. Interchanging symbols, writing z for z_0 and vice versa,

$$v_0 = \pm \int_{-1}^{+1} \frac{1}{\pi} \frac{u dz}{z - z_0} \sqrt{\frac{1 - z^2}{1 - z_0^2}} \tag{16}$$

For a point near the edge on the positive $z_0 = 1$, write

$$z_0 = 1 - \epsilon$$

$$v_{edge} = -\frac{1}{\pi\sqrt{2\epsilon}} \int_{-1}^{+1} u dz \sqrt{\frac{1 + z}{1 - z}}$$

or, substituting $\sqrt{2\epsilon} = \sin \delta_{edge}$

$$v_{edge} = -\frac{1}{\pi(\sin\delta)_{\delta=0}} \int_{-1}^{+1} u dz \sqrt{\frac{1 + z}{1 - z}} \tag{17}$$

$\sin \delta_{edge}$ approaches zero, hence $v_{edge} = \infty$.

For the application to the elements of the wing theory, in addition to the flows mentioned, there is one flow which needs a discussion of its own. This is given by the potential function

$$F = A_0 \sin^{-1} z = -A_0 \delta \tag{18}$$

The velocity function of this flow is

$$F' = \frac{A_0}{\sqrt{1 - z^2}} = -\frac{A_0}{\sin\delta} \tag{19}$$

We shall call this flow "circulation flow" as it represents a circulation of the air around the line. The transverse component of the velocity at points along the line is identically zero.

The circulation flow does not quite fit in with the other ones represented by equation (3), because the potential function (18) is a multiple valued one, the values at any one point differing by 2π or multiples thereof.

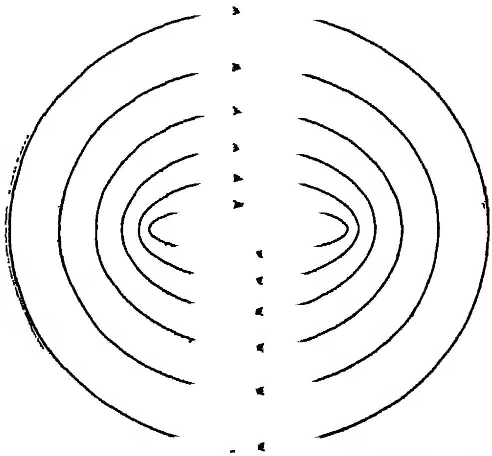


Figure 4. Circulation Flow Around the Straight Line

This is in accordance with the physical consideration, that it is impossible to produce this flow by an impulsive pressure over the straight line. Such a pressure would not perform any mechanical work, as the transverse components of velocity at points along the line are zero. The kinetic energy of this flow, on the other hand, is infinite, and hence this flow cannot even be completely realized. Still it plays the most important part in aerodynamics.

The best way to understand this flow and its physical meaning is to suppose the line to be elongated at one end, out to infinity. On the one side the potential may be considered zero. Then it is constant and will be equal to 2π on the other side. The transverse velocity component is finite. Hence, the flow can be produced by a constant impulsive pressure difference along this line extending from the edge $z = 1$ to infinity. This pressure difference makes the fluid circulate around the original straight line, the pressure along the line itself being given by the potential function (18) and not performing any work.

A pressure difference along an infinite line does never actually occur. At least it does not occur simultaneously along the whole line. A very similar thing, however, occurs very often which has the same effect. That is a constant momentum being transferred to the air at right angles to an infinite straight line at one point only, but the point traveling along the line, so that the final effect is the same as if it had occurred simultaneously. This is fundamentally the case of an airplane flying along that infinitely long line. During the unit of time it may cover the length V and transfer to the air the momentum L, equal to the lift of the airplane. Then the impulse of the force, per unit of length of the line, is L/V, and hence the potential difference is $L/V\rho$. That makes A_0 in equation (18) $A_0 = L/zV\rho\pi$. If the airplane has traveled far enough, the flow in the neighborhood of the wing, or rather one term of the flow, is described by the circulation flow, provided that the airplane is two-dimensional, that is, has an infinite span.

The velocity at the end of the wing $z = +1$ due to this circulation flow

$$F = -A_0\delta$$

where
$$A_0 = L/2\pi\rho V \tag{20}$$

is
$$V_{edge} = -\left(\frac{A_0}{\sin\delta}\right)_{\delta=0} = \frac{L}{2\pi\rho V(\sin\delta)_{\delta=0}} \tag{21}$$

25. Synopsis

Let there be a straight line of the length $2a$, and introduce δ by means of $\cos \delta = x/a$, where x is the distance of a point at the line from its center.

The two flows of most special interest are the circulation flow and the transverse flow. The former, at the points of the line, has the potential $A\delta$, the transverse velocity component zero, and the longitudinal velocity component $A/\sin \delta$, where A is an arbitrary constant. The transverse flow has the potential $A \sin \delta$, the constant transverse velocity A, and the longitudinal velocity component $A \cos \delta/\sin \delta$.

In general, the potential can be $A_n \sin n\delta$, the transverse velocity component $A_n n \sin n\delta/\sin\delta$, and the longitudinal velocity component $A_n n \cos n\delta/\sin\delta$, where A_n is an arbitrary constant and n an arbitrary integer.

These general flows with all values of n can be combined to a Fourier's series. There are three series, respectively representing the potential, and the transverse and longitudinal velocity components along the line. Each can be used for the determination of the two others. Determine the constants A_n of the series for the given function by means of the integration theorem of the Fourier's series. Insert the same constants into the two other series, and sum up the series.

If the transverse velocity is given, the longitudinal velocity can also be determined by means of the definite integral (15).

The line traveling with the velocity V and with a circulation flow $A_0\delta$ experiences a lift $L = 2\pi A_0 V\rho$, where ρ denotes the density of the fluid.

26. Problems and Suggestions

1. The transverse velocity component at all points of a straight line has the velocity ½ ft./sec. What is the magnitude of the longitudinal velocity component one-quarter of the length from the end?

2. The longitudinal velocity is $\cos \delta/\tan \delta - \sin \delta$, where

the length of the line is 2 and δ defined as in the text. How large is the transverse velocity and the potential?

3. The potential in the last problem is the integral of the longitudinal velocity with respect to the length x. Is that always so?

4. The transverse velocity of the line in Problem 2 is $\cos \delta$. Compute the longitudinal velocity by means of Fourier's series. The integral (15) gives the same solution.

5. A rigid straight line moves transversely with a velocity of 30 ft./sec.; how large is the difference of pressure between the points 10% and 50% from the edge, if the density of the perfect fluid is 1/420 lb. sec^2/ft^4?

6. The rigid straight line is rotating about its center with the velocity of the ends 30 ft./sec. How large is the longitudinal velocity at the center?

7. How large is the infinite edge velocity of the same line, expressed as a multiple of $1/\sin \delta$?

8. The same line performs the same rotation and at the same time the center moves transversely. How large has this transverse velocity to be in order to obtain a finite edge velocity at one end?

9. Around which of its points has a straight line to be turned in a perfect fluid in order to have an infinite velocity at one end only?

10. How large is the kinetic energy of the flow caused by the rotating line in Problem 6? How large the apparent moment of inertia?

11. Describe the velocity distribution represented by the complex potential function $\log z$.

12. Do the same for the potential function $i \log z$.

13. The transverse velocity of a straight line of the length l is one. Which is the circulation flow leaving one edge velocity finite?

CHAPTER IV

THEORY OF THE WING SECTION

27. Outlook

We determine the two-dimensional potential flow produced by the motion of a wing section by replacing the section by its mean curve of zero thickness. The potential flow is then computed for a line which, although straight, is in the process of deflecting itself so that the transverse velocity components are the same as with the rigid mean curve of the wing section.

A circulation flow is superimposed strong enough to obtain finite velocity of flow around the rear edge.

We compute the longitudinal velocity components. From them we compute the pressures at all points of the wing section, and they at last give the lift and moment.

The lift and moment are expressed by introducing the magnitude of the angle of attack of a straight line of the same chord, giving the same lift or moment.

Rules for the convenient determination of these angles, either numerically or graphically, are given.

28. Representation of the Flow

The investigation of the air flow around wings is of great practical importance in view of the predominance of heavier-than-air craft. It is necessary to divide this problem into two parts, the consideration of the cross-section of one or several wings in a two-dimensional flow, and the investigation of the remaining effect. This chapter is devoted to the first question.

All wings in practice have a more or less rounded leading edge, a sharp trailing edge and the section is rather elongated, being as a first approximation described by a straight line. The appli-

cation of the aerodynamic flows around a straight line for the investigation of the flow around a wing section suggests itself. We saw in the last chapter how the potential flow around a straight line is determined, for instance, from the transverse components of velocity along this line. Only one type of flow, the circulation flow, is excepted. This flow does not possess any transverse components at the points of the line and hence can be superposed on a potential flow of any magnitude without interfering with any prescribed transverse velocity. We saw, on the other hand, that it is just this circulation flow not determined so far, which gives rise to the chief quantity, the lift. It is, therefore, necessary to find some additional condition for determining the magnitude of the circulation flow.

Figure 5 Flow Conforming with Kutta's Condition

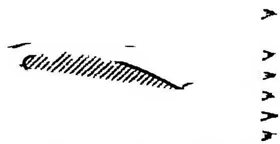


Figure 6. Flow NOT Conforming with Kutta's Condition

This magnitude of the circulation flow is physically determined by the fact that the air is viscous, no matter how slightly viscous it is. The additional condition governing the magnitude of the circulation flow can be expressed without any reference to the viscosity and was done so in a very simple way by Kutta.[1] The condition is very plausible, too. Kutta's condition simply states that the air does not flow with infinite velocity at the sharp, rear edge of the wing section. On the contrary, the circulation flow assumes such strength that the air leaves the section exactly at its rear edge, flowing there along the section parallel to its mean direction. The wing thus acts as a device forcing the air to leave the wing flowing in a particular direction.

Consider, for instance, the wing section which consists merely of a straight line of the length 2. The angle of attack may be

[1] See references 6, 7, and 8, page 175.

α. The flow produced by this line moving with the velocity V is the transverse flow with the constant transverse component of velocity along the wing V sin α. The infinite longitudinal velocity at the rear end is

$$-V \sin\alpha \cdot \frac{1}{(\sin\delta)_{\delta=0}}$$

Since the angle of attack may be assumed to be small, we can change slightly the way of representing the flow, by turning the real axis of coordinates into the direction of motion. Instead of referring the flow to the line really representing the wing section, we consider a straight line parallel to the motion, and having the same length as the wing. Since this line differs only slightly from the wing, the transverse components of the flow relative to this line are approximately equal to the transverse

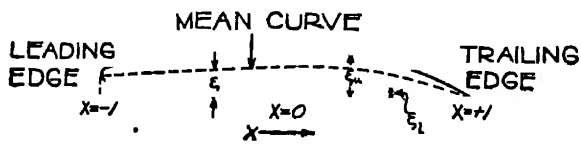


Figure 7. Mean Curve of a Wing Section

velocity relative to the wing section at the nearest point and, therefore, are constant again and equal to V sin α. Thus this approximate representation of the flow is equivalent to the more exact way given before. It also gives the same infinite velocity at the rear end.

This velocity determines the magnitude of the circulation flow

$$F = A_0 \sin^{-1} z \qquad \text{(Eq. 18, Chap. III)}$$

by the condition that the sum of their infinite velocities at the edge is zero.

$$V \sin\alpha \frac{1}{(\sin\delta)_{\delta=0}} - A_0 \frac{1}{(\sin\delta)_{\delta=0}} = 0$$

and hence $A_0 = V \sin\alpha$. The lift is, therefore

$$L = 2\pi A_0 V\rho = 2\pi V^2 \rho \sin\alpha$$

The lift coefficient, defined by $C_L = \dfrac{L}{SV^2\rho/2}$, since the chord = 2, is therefore

$$C_L = 2\pi \sin\alpha, \text{ or approximately } 2\pi\alpha \tag{22}$$

and

$$L = V^2\rho/2 \cdot S \cdot 2\pi\alpha \tag{23}$$

where S denotes the area of the wing.

29. Computation of the Lift

The representation of the flow just employed is approximately correct and gives the same result as the exact method. This new method now can be generalized so that the lift of any wing section, other than a straight line, can be computed in the same way, too. The section can be replaced with respect to the aerodynamic effect by a mean curve, situated in the middle between the upper and lower curves of the section and having at all points the same mean direction as the portion of the wing section represented by it. The ordinates of this mean wing curve may be ξ, the abscissa x, so that the direction of the curve at each point is $d\xi/dx$. This direction can be considered as the local angle of attack of the wing, identifying the sine and tangent of the angle with the angle itself. Accordingly, it is variable along the section. Since the velocity of the air relative to the wing is approximately equal to the velocity of flight, the component at right angles to the x axis is $Vd\xi/dx$. As before, the infinite velocity at the rear edge is to be found. It is, according to equation (17)

$$\frac{-V}{\pi(\sin\delta)_{\delta=0}} \int_{-1}^{+1} \frac{d\xi}{dx} \sqrt{\frac{1+x}{1-x}}\,dx \tag{24}$$

At the rear edge $x = 1$. The equivalent angle of attack, that is the angle of attack of the straight line giving the same lift as the wing section, is found by the condition that this infinite

value must be the same as that deduced for a straight line; viz., $V \sin\alpha/\sin\delta$. Hence, replacing $\sin\alpha$ by α,

$$\alpha' = -\frac{1}{\pi}\int_{-1}^{+1}\frac{d\xi}{dx}\sqrt{\frac{1+x}{1-x}}dx \qquad \text{length} = 2 \qquad (25)$$

$$\alpha' = -\frac{2}{\pi}\int_{-1/2}^{+1/2}\frac{d\xi}{dx}\sqrt{\frac{1+2x}{1-2x}}dx \qquad \text{length} = 1 \qquad (25a)$$

This formula holds true for any small angle of attack of the section.

The integral can now be transformed into one containing the coordinate ξ rather than the inclination $d\xi/dx$ of the wing curve, provided that the trailing edge is situated at the x axis, that is, if ξ is zero at the end $x = +1$. This transformation is performed by integration by parts, considering $d\xi/dx \cdot dx$ as a factor to be integrated. It results

$$\alpha' = \frac{1}{\pi}\int_{-1}^{+1}\frac{\xi dx}{(1-x)\sqrt{1-x^2}} \qquad \text{length} = 2 \qquad (26)$$

$$\alpha' = \frac{4}{\pi}\int_{-1/2}^{+1/2}\frac{\xi dx}{(1-2x)\sqrt{1-4x^2}} \qquad \text{length} = 1 \qquad (26a)$$

The important formula (26) gives the equivalent angle of attack directly from the coordinates of the shape of the wing section. The mean height ξ of the section has to be integrated along the chord after having been multiplied by a function of the distance from the leading edge, the same one for all wing sections. This integration can always be performed, whether the section is given by an analytical expression, graphically, or by a table of the coordinates. In the latter case a numerical integration can be performed.

30. Computation of the Moment

The lift of a wing section as computed in Section 29 is caused by the circulation flow symmetrical with respect to the straight line representing the wing. Hence, the pressure creating this lift is located symmetrically to the wing, its center of pressure is at 50% of the chord, it produces no moment with respect to the middle of the wing. This lift is not, however, the entire resultant air force. The remaining aerodynamic flow in general exerts a resultant moment (couple of forces) and this moment removes the center of pressure from its position at 50%.

If the wing section is a straight line of the length 2, its apparent transverse mass is $\pi\rho$ as seen in Section 24. The longitudinal mass is zero. Hence, according to equation (2), Section 12, the resultant moment is

$$M = V^2\rho/2 \cdot \pi \sin 2\alpha \qquad \text{length} = 2 \tag{27}$$

$$M = V^2\rho/2 \cdot 2\pi\alpha \qquad \text{length} = 2 \tag{28}$$

Both the exact and the approximate expressions give the constant center of pressure 25% of the chord from the leading edge, which results from dividing the moment by the lift (23).

The straight sections considered have a constant center of pressure independent of the angle of attack. The center of pressure does not travel. This is approximately true also for symmetrical sections with equal upper and lower curves, where the center of pressure is also at 25%. If, however, the upper and lower curves are different and hence the mean section curve is no longer a straight line, the potential flow produced at the angle of attack zero of the chord not only gives rise to the circulation flow and thus indirectly to a lift, but also creates a moment of its own. It is simple to compute this moment from the potential flow, which is represented in equation (8) as a superposition of the flows, equation (3).

The longitudinal velocity relative to the line is, according to equation (6),

$$v = \left(A_1\frac{\cos\delta}{\sin\delta} + A_2\frac{2\cos 2\delta}{\sin\delta} + \ldots + A_n\frac{n\cos n\delta}{\sin\delta} + \ldots\right) + V$$

As the section is supposed to be only slightly curved, $d\xi/dx$ is always small, therefore; so are the coefficients A_n when compared to V, so that they may be neglected when added to it. The pressure at each point along the line according to Bernouilli's theorem is:

$$p = -\rho/2 \cdot v^2$$

The present object is the computation of the resultant moment. When really forming the square of the bracket in the last expression, the term with V^2 indicates a constant pressure and does not give any resultant moment. The squares of the other terms are too small and can be neglected. There remains only the pressure,

$$p = -\rho V\left(A_1\frac{\cos\delta}{\sin\delta} + A_2\frac{2\cos 2\delta}{\sin\delta} + \ldots\right)$$

giving the resultant moment about the origin

$$M = 2\int_{-1}^{+1} pxdx,$$

or

$$M = 2V\rho\int_0^{\pi}\left(A_1\frac{\cos\delta}{\sin\delta} + A_2\frac{2\cos 2\delta}{\sin\delta} + \ldots\right)\cos\delta\,\sin\delta d\delta$$

taken twice on one side only, since the density of lift is twice the density of pressure, the pressure being equal and opposite on both sides of the wing. But

$$\int_0^{\pi}\cos n\delta\,\cos m\delta d\delta = 0$$

if m and n are different integers. Hence there remains only one term. The resultant moment is

$$M = 2\rho\, VA_1\,\pi/2$$

A_1 was found according to equation (11) by means of the integral

$$A_1 = \frac{2}{\pi}\int_0^\pi V\frac{d\xi}{dx}\sin^2\delta d\delta$$

Hence the moment is

$$M = 2\rho V^2\int_0^\pi \frac{d\xi}{dx}\sin^2\delta d\delta$$

or, expressed by x

$$M = -2\rho V^2\int_{-1}^{+1}\frac{d\xi}{dx}\sqrt{1-x^2}dx \qquad (29)$$

By the same method as used with integral (25) this integral can be transformed into

$$M = -2\rho V^2\int_{-1}^{+1}\frac{\xi x dx}{\sqrt{1-x^2}} \qquad (30)$$

It has been shown that for a straight line of length 2, the center of pressure has a lever arm ½ and the lift is $V^2\rho/2 \cdot 2\pi\alpha \cdot 2$, giving a moment $V^2\rho\pi\alpha$. Consequently, the resultant moment is the same as if the angle of attack is increased to the equivalent angle for moment α''.

$$\alpha'' = -\frac{2}{\pi}\int_{-1}^{+1}\frac{\xi x dx}{\sqrt{1-x^2}} \qquad \text{length} = 2 \qquad (31)$$

$$\alpha'' = -\frac{16}{\pi}\int_{-1/2}^{+1/2}\frac{\xi x dx}{\sqrt{1-4x^2}} \qquad \text{length} = 1 \qquad (32)$$

It is readily seen that this angle is equal to angle of attack of the chord for sections with section curves equal in front and in rear. Hence, such sections have the center of pressure at

50% for the angle of attack zero of the mean curve, that is, for the lift (24) produced by the shape of the section only. The additional lift produced at any other angle of attack of the chord and equal to the lift as produced by the straight line at that angle of attack, has the center of pressure at 25%. Hence, a travel of the center of pressure takes place toward the leading edge when the angle of attack is increased, approaching the point 25% without ever reaching it. The same thing happens for other sections with the usual shape. At the angle of attack zero of the chord the lift produced was seen to be $2\pi V^2\rho\alpha'$, i.e., from (26)

$$L = V^2\frac{\rho}{2}4\int_{-1}^{+1}\frac{\xi dx}{(1-x)\sqrt{1-x^2}}$$

and the moment,

$$M = -2\rho V^2\int_{-1}^{+1}\frac{x\xi dx}{\sqrt{1-x^2}} \tag{30}$$

giving the center of pressure at the distance from the middle

$$\frac{\displaystyle\int_{-1}^{+1}\frac{x\xi dx}{\sqrt{1-x^2}}}{\displaystyle\int_{-1}^{+1}\frac{\xi dx}{(1-x)\sqrt{1-x^2}}} \qquad \text{length} = 2$$

Its distance from the leading edge, in per cent of the chord, is

$$CP = 50\%\left[1+\frac{\displaystyle\int_{-1}^{+1}\frac{x\xi dx}{\sqrt{1-x^2}}}{\displaystyle\int_{-1}^{+1}\frac{\xi dx}{(1-x)\sqrt{1-x^2}}}\right]$$

The moment with respect to the point 25% from the leading edge is independent of the angle of attack. It is

$$M_{25\%} = -4V^2\frac{\rho}{2}\left[\int_{-1}^{+1}\frac{x\xi dx}{\sqrt{1-x^2}} + \frac{1}{2}\int_{-1}^{+1}\frac{\xi dx}{(1-x)\sqrt{1-x^2}}\right] \quad \text{length} = 2$$

The moment coefficient defined $C_m = \dfrac{M}{bc^2V^2\rho/2}$ is then

$$C_{m_{25\%}} = \int_{-1}^{+1}\frac{x\xi dx}{\sqrt{1-x^2}} + \frac{1}{2}\int_{-1}^{+1}\frac{\xi dx}{(1-x)\sqrt{1-x^2}} \tag{33}$$

We have thus obtained expressions for the two chief air force coefficients of wings.

31. Computation in Special Cases

We apply formula (26) for the equivalent angle of attack with respect to lift to a wing section represented by a straight line

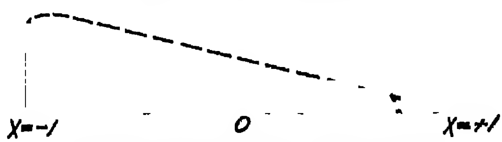


Figure 8. Straight Mean Wing Curve

inclined under the angle α, in order to illustrate its use and to obtain a check on the formula: for the equivalent angle of such line is, of course, α. Accordingly, let

$$\xi = \alpha(1-x)$$

$$\alpha_L = \frac{\alpha}{\pi}\int_{-1}^{+1}\frac{(1-x)dx}{(1-x)\sqrt{1-x^2}} = \frac{\alpha}{\pi}\Big|_{-1}^{+1}\sin^{-1}x\Big| = \alpha$$

as it should be.

We apply next formula (26) to a section represented by a parabolic arc,

$$\xi = a(1 - x^2)$$

$$\alpha_L = \frac{a}{\pi}\int_{-1}^{+1}\frac{1 - x^2}{1 - x}\,\frac{dx}{\sqrt{1 - x^2}} = \frac{a}{\pi}\int_{-1}^{+1}\frac{1 + x}{\sqrt{1 - x^2}}\,dx$$

$$= \frac{a}{\pi}\Big|\sin^{-1}x - \sqrt{1 - x^2}\Big|_{-1}^{+1} = a$$

The computation shows the straight line, creating the same lift, to be parallel to the line connecting the trailing edge with the peak of the arc.

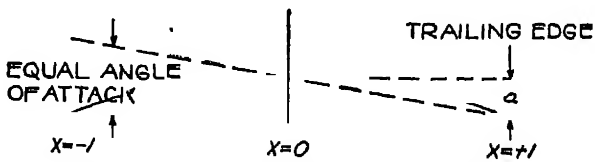


Figure 9. Parabolic Arc as Mean Wing Curve

We compute at last the effect of displacing the elevator by the angle β. Let the length be 2 again and let a denote the

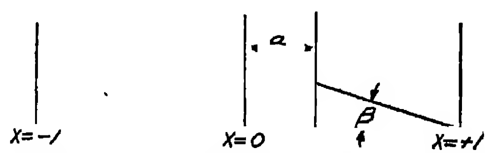


Figure 10. Mean Curve of a Displaced Elevator

distance of the hinge from the center of the chord of the entire tail-plane combination. The section of the combination, having been straight before the elevator was displaced, after the displacement is represented by

$$\xi = (1 - a)\beta$$

between the points -1 and a, and $\xi = (1 - x)\beta$ between the points a and 1. This substituted into formula (26) gives

$$\alpha_0 = \frac{1}{\pi}\int_{-1}^{+1} \frac{\beta(1-x)dx}{(1-x)\sqrt{1-x^2}} - \frac{1}{\pi}\int_{a}^{-1} \beta\frac{a-x}{1-x}\frac{dx}{\sqrt{1-x^2}}$$

The first integral is the same as in the first example and was there found to give β. For the remaining integral we write

$$\frac{1}{\pi}\int_{-1}^{a} \beta\frac{a-x}{1-x}\frac{dx}{\sqrt{1-x^2}} = \frac{\beta}{\pi}\int_{-1}^{a}\left(\frac{1-x}{1-x}\frac{dx}{\sqrt{1-x^2}} + \frac{a-1}{1-x}\frac{dx}{\sqrt{1-x^2}}\right)$$

$$= \frac{\beta}{\pi}\left| \sin^{-1}x + (a-1)\frac{1+x}{\sqrt{1-x^2}} \right|_{-1}^{a} = \frac{\beta}{\pi}\left(\sin^{-1}a + \frac{\pi}{2} - \sqrt{1-a^2}\right)$$

Hence, both integrals together give

$$\alpha_0 = \beta\left(½ - \frac{\sin^{-1}a}{\pi} + \frac{\sqrt{1-a^2}}{\pi}\right)$$

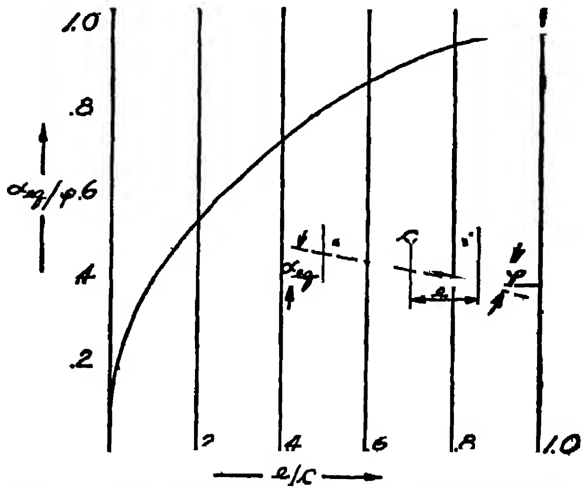


Figure 11. Equivalent Angle of Attack of a Displaced Elevator, as Fraction of the Displacement Angle, Plotted Against the Elevator Depth

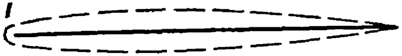
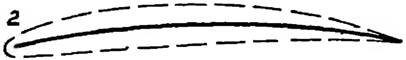
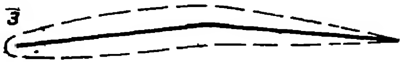
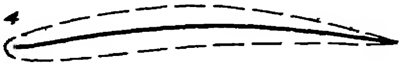
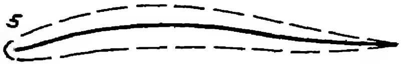
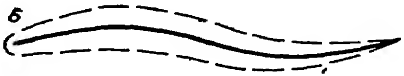
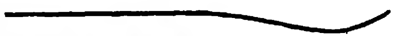
Shape.	Equation of shape.	α	C_{m_0}
1 	$\xi=0$	0	0
2 	$\xi=y(1-x^2)$	y	0
3 	$\xi=y(1\pm x)$	$y\cdot\frac{2}{\pi}$	0
4 	$\xi=y(1-x^2)^{3/2}$	$y\frac{2}{\pi}$	0
5 	$\xi=y(1+x)^{2.5}(1-x)^{1.5}$	$y\frac{8}{3\pi}$	$-\frac{16}{15}y$
6 	$\xi=-y(1-x^2)x$	$-\frac{y}{2}$	$-\frac{\pi}{2}y$
7 	0 between -1 and 0 $\xi=yx^2(1-x^2)$ between 0 and $+1$	$y\left(\frac{1}{4}-\frac{2}{3\pi}\right)$	$-\frac{2}{15}y$

Figure 12. Various Algebraic Mean Curves and Their Equivalent Angles

The value of the bracket is plotted in Figure 11 against the ratio of the elevator chord to the entire chord. As should be expected, it is less than one.

32. Numerical and Graphical Methods

In all practical cases the shape of the wing section is not given by an analytical expression, but either by a table of ordinates or directly by a drawing. We proceed to the problem, how then to determine the two equivalent angles of attack, and we begin with their computation from ordinates.

There are generally 15 to 20 upper and lower ordinates given, closer together near the leading edge. The evaluation of integral (26) by the ordinary numerical method, multiplying each mean of the 15 to 20 ordinates by $1/(1 - x)\sqrt{1 - x^2}$ and by the arithmetic mean of the two adjoining spaces, and then adding, is feasible. That would settle the question.

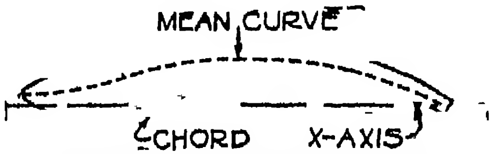


Figure 13. Axis for Lift Computation

It requires less time, however, to determine the mean ordinates at certain specified points, by interpolation, suitable for the application of Gauss' method of numerical integration in a modified form. We proceed to compute the necessary constants.

The integral (26) gives a finite value only if $\xi = 0$ at the point $x = -1$, that is, at the trailing edge. Hence, if the trailing edge is so thick that the rear end of the mean curve has an ordinate ξ of noticeable length, it is necessary to move the x axis so as to make $\xi = 0$ at this point (Figure 13). It is, however, sufficient to move the chord parallel to itself, that is, to diminish all ordinates ξ by the same amount.

Since ξ, which is now regarded as a function of x, becomes necessarily zero at the point $x = -1$, it can be written $\xi = (1 - x) F(x)$, where $F(x)$ is finite over the whole chord and is defined by

$$F(x) = \frac{\xi}{1 - x} \tag{34}$$

The integral (26) can now be written

$$\alpha_0' = \frac{1}{\pi}\int_{-1}^{+1}\frac{F(x)dx}{\sqrt{1 - x^2}} = \frac{1}{\pi}\int_{-\pi/2}^{+\pi/2} F(x)d\delta \tag{35}$$

where $x = \sin\delta$.

The problem is now much simplified, being reduced to the evaluation of a simple integral of a function, not of a product of two functions, within the range from $\delta = -\pi/2$ to $\delta = +\pi/2$. This function $F(\delta)$, it is true, is probably in most cases smaller near the leading edge than at the other parts of the intervals. But we will not take this into account. Then Gauss' method can be applied directly.

This method consists in selecting the values of the function to be integrated at certain points x, multiplying them by certain factors A and adding the products obtained. The points are so chosen that they give a result more exact in general than the result obtained from the same number of points otherwise located. If, for instance, only one point and the value of the function at this point ξ_0 shall be used for the integration, the best position of this point is in the middle of the interval, in our case $\delta = 0$; i.e., $x = 0$. This is indeed the only point which gives the result absolutely correct not only if $F(x)$ is constant and hence can be written $F(x) = \xi_0$, but also if it is any linear function of δ, $F(x) = \xi_0 + \beta\delta$, containing the given point. The result is found by substituting for $F(x)$ any constant function having the same value as F at the point considered, that is, $F = \xi_0$.

$$\alpha_0 = -\frac{1}{\pi}\int_{-\pi/2}^{+\pi/2}\xi_0 d\delta = -\xi_0$$

This refers to the length of the chord 2. For any length of the chord c, the results appear $\alpha_0 = -\frac{2\xi_0}{c}$ in radians. It results, therefore, that for the first provisional determination of the direction of zero lift, using only one point of the mean curve of the section, the middle of the section, 50% of the chord, is the most characteristic point. The zero lift direction is, as first approximation, parallel to the line connecting the trailing edge and the mean curve at 50% of the chord (Figure 14).

The wing sections used in practice are always smooth and regular, but still the use of one point only is often not exact enough. It is desirable to use at least two or three points. Even five points may sometimes be desired. For more than one point, Gauss computed a table, from which the following values are taken.

GAUSS' RULE

$$\int_{-1}^{+1} F(\bar{x})d\bar{x} = A_1F(\bar{x}_1) + A_2F(\bar{x}_2) + \ldots + A_nF(\bar{x}_n)$$

$n = 1$	$\bar{x}_1 = 0$	$\frac{1}{2}A_1 = 1$
$n = 2$	$\bar{x}_1 = -\bar{x}_2 = .577{,}350{,}2691$	$\frac{1}{2}A_1 = \frac{1}{2}A_2 = \frac{1}{2}$
$n = 3$	$\bar{x}_1 = -\bar{x}_3 = .774{,}596{,}6669$	$\frac{1}{2}A_1 = \frac{1}{2}A_3 = 5/18$
	$\bar{x}_2 = 0$	$\frac{1}{2}A_2 = 4/9$
$n = 5$	$\bar{x}_1 = -\bar{x}_5 = .906{,}179{,}8459$	$\frac{1}{2}A_1 = \frac{1}{2}A_5 = .118{,}463{,}4425$
	$\bar{x}_2 = -\bar{x}_4 = .538{,}469{,}3101$	$\frac{1}{2}A_2 = \frac{1}{2}A_4 = .239{,}314{,}3352$
	$\bar{x}_3 = 0$	$\frac{1}{2}A_3 = .284{,}444{,}4444$

The substitution of the expression (35) into Gauss' rules, x expressed in per cent of the chord, and the resulting equivalent angle of attack expressed in degrees, gives the following table. The equations used for its computation are

$$x_n = 50\left[1 - \sin\left(\bar{x}_n \cdot \frac{\pi}{2}\right)\right]; \quad f_n = \frac{90}{\pi} \cdot \frac{A_n}{1 - x}$$

The use of two points is exact enough in most cases. Obtain the ordinates at the points x% of the chord, multiply them by the

value of f, belonging to the point and add. That gives directly the equivalent angle of attack in degrees.

EQUIVALENT ANGLE FOR LIFT IN DEGREES

$$\alpha_0 = f_1\, \xi_1/c + f_2\, \xi_2/c + f_n\, \xi_n/c$$

$n = 1$	$x_1 = 50\%$	$f_1 = 114.6$
$n = 2$	$x_1 = 89.185\%$	$f_1 = 264\ 9$
	$x_2 = 10.815\%$	$f_2 = 32.12$
$n = 2$	less exact:	
	$x_1 = 90\%$	$f_1 = 286$
	$x_2 = 10\%$	$f_2 = 31.9$
$n = 3$	$x_1 = 96.90\%$	$f_1 = 513.08$
	$x_2 = 50\%$	$f_2 = 50,93$
	$x_3 = 3\ 10\%$	$f_3 = 16.425$
$n = 5$	$x_1 = 99.458\%$	$f_1 = 1,252\ 24$
	$x_2 = 87.426\%$	$f_2 = 109.048$
	$x_3 = 50\%$	$f_3 = 32.5959$
	$x_4 = 12.574\%$	$f_4 = 15.6838$
	$x_5 = .542\%$	$f_5 = 5.97817$

$100\% =$ trailing edge
$c =$ length of chord

We proceed now to the computation of the equivalent angle for the moment around the origin; that is, 50% of the wing chord. The integral for the computation of this angle is:

$$\alpha_0'' = -\frac{2}{\pi}\int_{-1}^{+1} \frac{\xi}{\sqrt{1 - x^2}}\, dx \tag{31}$$

This integral converges for any finite ξ, so it is not necessary for its evaluation that the trailing edge coincide with the x axis. The conventional chord can always be used without any correction for the thickness of the trailing edge.

A closer examination of integral (31) shows that the factor of ξ is an odd function having opposite values for pairs of points at equal distance from the middle of the chord. It is, therefore, at once obvious that a symmetrical curve has the same equiv-

alent angle of attack for moment as its chord. Only an unsymmetric curve differently shaped at the front and at the rear part gives an angle different therefrom. The degree of dissymmetry cannot be derived from one ordinate ξ only, at least a pair is required On the other hand, one pair is sufficient for most practical purposes.

Write $\xi = \xi_0 + xF(x)$

where ξ_0 is the value of ξ at $x = 0$. Then

$$\alpha_0' = -\frac{2}{\pi}\int_{-1}^{+1} \frac{F(x)x^2}{\sqrt{1-x^2}}dx = \frac{1}{\pi}\int_{-1}^{+1} F(x)\,d\,[\sin^{-1}x - x\sqrt{1-x^2}]$$

$$= \frac{2}{\pi}\int_{0}^{1} [F(x) - F(-x)]\,d\,[\sin^{-1}x - x\sqrt{1-x^2}]$$

The interval of (arc sin $x - x\ \sqrt{1-x^2}$) extends between 0 and $\frac{1}{2}\pi$. Hence the best ordinate to be chosen for the computation has the abscissa (arc sin $x - x\ \sqrt{1-x^2}) = \frac{1}{4}\pi$.

Write $x = \sin\delta$, then the condition is

$$2\delta - \sin 2\delta = \frac{\pi}{2};\ \left(\frac{\pi}{2} - 2\delta\right) + \cos\left(\frac{\pi}{2} - 2\delta\right) = 0$$

This equation has the solution $\delta = 66°\ 10.4'$.

$$x = \sin\delta = .91476 = 4.26\% \text{ and } 95.74\%$$

The factor for these ordinates is easily found from the consideration that the straight line gives the exact direction, coinciding with its direction. The distance between the points is $c(95.74\% - 4.26\%)$; the difference between the ordinates is $\xi_1 - \xi_2$, hence the angle between the x axis and the straight line connecting the two points in degrees is

$$\frac{180}{\pi}\ \frac{\xi_1 - \xi_2}{c(.9574 - .0426)} = 62.634\ \frac{\xi_1 - \xi_2}{c}$$

Hence the factor is

$$f = \frac{\pm 180}{.91476 \cdot \pi} = 62.634$$

$$x_1 = 4.26\% \quad x_2 = 95.74\%$$

These factors are used for the computation of the equivalent angle of attack for the moment, in the same way as the constant, in the last table.

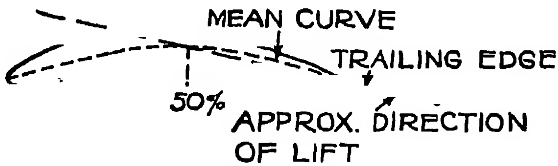


Figure 14. Approximate Angle of Attack for Lift

The computation of the equivalent angles of attack can easily be performed graphically, using the results just obtained.

It has already been mentioned that the line connecting the rear edge with the point at 50% of the mean curve gives the first approximation of the equivalent angle of attack for the lift.

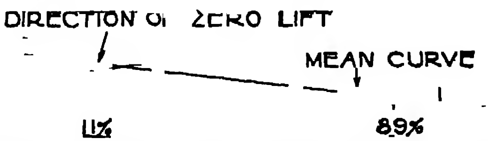


Figure 15. Determination of Angle of Attack for Lift

A better approximation is obtained by connecting the rear edge with the two points at 89% and 11% chord at the mean curve and by bisecting the angle formed by these two connection lines.

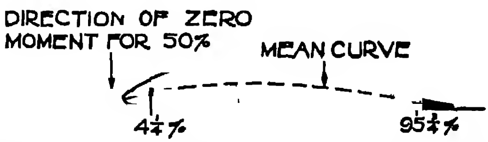


Figure 16. Angle of Attack for Moment Around the Center

Figure 17. Wing Section with Fixed Center of Pressure

The equivalent angle of attack for the moment about 50% of the chord is obtained by connecting the two points at 4¼% and 95 ¾% chord at the mean curve.

33. Remarks on the Theory of Multiplane Wing Sections

The problem of two or even more wing sections, combined to a biplane or multiplane surrounded by a two-dimensional flow, can be treated in the same way as the single wings. The two sections determine by their slope at each point a distribution of transverse components along parallel lines. The distribution determines a potential flow with resulting moment. Kutta's condition of finite velocity near the two rear edges determines an additional circulation flow giving rise to a lift and moment. The aspect of the question offers nothing new, it is a purely mathematical problem.

This mathematical problem has not yet been solved in this extension. The author has attacked the problem within a more narrow scope (see No. 22 author's papers listed in Appendix). The method followed there amounts to the following considerations:

Equation (10) represents different types of flow around one straight line, consisting in a motion of the air in the vicinity of the straight line only. Now the motion of the flows with high order n is more concentrated in the immediate neighborhood of the straight line than the flows of low order n. The transverse velocity components along the line, determining the flow, change their sign $(n - 1)$ times along the line. With large n, positive and negative components follow each other in succession very rapidly so that their effect is neutralized even at a moderate distance.

Hence, the types of flow of high order n around each of a pair

of lines will practically be the same as if each line is single. The flows of high order do not interfere with those of a second line in the vicinity even if the distance of this second line is only moderate. It will chiefly be the types of flow of low order, the circulation flow $n = 0$, the transverse flow $n = 1$, or it may be the next type $n = 2$ which differ distinctly whether the wing is single or in the vicinity of a second wing. Accordingly, I computed only the flows of the order $n = 0$ and $n = 1$, the circulation flow and the transverse flow for the biplane and used the other flows as found for the single section.

The results are particularly interesting for biplanes with equal and parallel wings without stagger. Their lift is always diminished when compared with the sum of the lifts produced by the two wings when single. The interference is not always the same. When the sum of the angle of attack and the mean apparent angle of attack with respect to the moment is zero, or otherwise expressed, at the angle of attack where the center of pressure is at 50%, it is particularly small. The lift produced at the angle of attack zero is diminished only about half as much as the remaining part of the lift produced by an increase of this angle of attack.

This second part of the lift does not have its point of application exactly at 25% of the chord, although its center of pressure is constant, too. This latter is quite generally valid for any two-dimensional flow. At any angle of attack zero arbitrarily chosen, the configuration of wing sections produces a certain lift acting at a certain center. The increase of the angle of attack produces another lift acting at another fixed point. Hence, the moment around this second center of pressure does not depend on the angle of attack; and the center of pressure at any angle of attack can easily be computed if the two centers of pressures and the two parts of the lift are known.

The resultant moment of the unstaggered biplane consisting of portions of equal and parallel straight lines is again proportional to the apparent transverse mass, as the longitudinal

mass is zero (refer to Chapter II). This mass is of use for the considerations of the next chapter also. If the two straight lines are very close together, the flow around them is the same as around a line of finite thickness and is almost the same as around one straight line. Its apparent mass is the same also, but in addition, there is the mass of the air inclosed in the space between the two lines and practically moving with them. Hence, the mass is approximately,

$$b\left(b\frac{\pi}{4} + h\right)\rho$$

where b is the length of the lines and h their distance apart, if the distance h of the lines is small. For great distance, on the other hand, the flow around each of the lines is undisturbed, the apparent mass is twice that of the flow around each line if single. It is, therefore,

$$2b^2\pi/4 \cdot \rho$$

For intermediate cases the apparent mass must be computed. Particulars on this computation are given in No. 37 of the author's papers (see Appendix). Table II gives the ratio of the apparent mass of a pair of lines to that of one single line for different values of h/b. This ratio, of course, is always between 1 and 2.

TABLE II. APPARENT MASS OF A PAIR OF STRAIGHT LINES

$\frac{\text{Gap}}{\text{Span}}$	$K = b^2\frac{\pi}{4} \cdot c$ c	$k = \frac{1}{\sqrt{c}}$
0.00	1.000	1.000
.05	1.123	.962
.10	1.212	.909
.15	1.289	.881
.20	1.353	.860
.30	1.462	.827
.40	1.550	.803
.50	1.626	.784
∞	2.000	.707

The theory of multiplane wings, on account of the elaborate mathematics required, is out of place in a book intended for engineers. The lift is smaller than with the same section used by itself in a monoplane, and the travel of the center of pressure is similar.

34. Synopsis

Let the mean curve of the wing section extend between the points 1 and -1. The equivalent angle of attack for lift is:

$$\alpha' = \frac{1}{\pi}\int_{-1}^{+1} \frac{\xi dx}{(1-x)\sqrt{1-x^2}} \text{ radians} \quad \text{length } 2 \tag{26}$$

The lift of the straight line inclined under the angle α' radians is:

$$L = V^2\, \rho/2 \cdot S \cdot 2\pi\alpha'$$

The equivalent angle of attack for the moment around the center is:

$$\alpha'' = -\frac{2}{\pi}\int_{-1}^{+1} \frac{\xi x dx}{\sqrt{1-x^2}} \text{ radians} \quad \text{length } 2 \tag{31}$$

The moment of the straight line inclined under the angle α'' radians, with respect to the center of the chord is:

$$M = V^2\, \rho/2 \cdot S \cdot c\, \pi/2 \cdot \alpha''$$

The moment of any section with respect to the point 25% from the leading edge is independent of the angle of attack and is:

$$M = V^2\, \rho/2 \cdot S \cdot c\, \pi/2\, (\alpha'' - \alpha')$$

These quantities can be determined numerically and graphically as described in Section 32.

35. Problems and Suggestions

The fluid is perfect, and the flow two-dimensional.

1. The maximum ordinate of a parabolic wing section is 8%

of its chord. At which angle of attack does the resultant air force pass through its rear edge?

2. How large is the lift coefficient in Problem 1, if the wing moves parallel to the chord?

3. The chord length of the straight stabilizer is two-thirds of the entire length of the chords of stabilizer and elevator. The angle of attack of the stabilizer is 5°. How much must the elevator be displaced to make the lift zero?

4. A wing section extending from -1 to $+1$ is S-shaped, according to the equation:

$$\xi = 0.05\, x \sqrt{1 - x^2}$$

so that the shape is equal at front and rear. Is the lift positive or negative if the motion is parallel to the line connecting the two ends of the section?

5. How large is the lift coefficient in the last problem if the wing moves with the angle of attack 5° of the chord connecting the ends?

6. How does the result of the graphical method compare with that of the analytical for the last problem?

7. Use the graphical method for the section in Problem 4 for the determination of the equivalent angle of attack for the moment.

8. Apply the graphical method with two points to the wing section in Figure 7.

9. How large is the lift of a symmetrical section (equal upper and lower curve) with the chord 90 cm., the angle of attack 6°, the velocity of flight 30 m./sec. and the density of air 1/8 kg. sec^2/m^4?

10. Under which angle of attack has the flow of the S-shaped wing section in Problem 4 no shock at the leading edge, that means a finite velocity at both ends?

11. The straight elevator is parallel to the direction of flight, but the leading edge of the straight stabilizer is inclined downwards. Which is the direction of lift?

12. The lift of any section remains unchanged if all abscissae parallel to the direction of flight are doubled and the ordinates left unchanged.

13. How large is the lift coefficient of a straight line moving parallel to the direction of motion with the velocity V and at the same time turning slowly about its center with the angular velocity ω?

14. The lift produced by the displacement of the elevator acts chiefly on the rigid portion of the control surface.

CHAPTER V

THEORY OF THE COMPLETE WING

36. Outlook

With actual wings, the air flow around each wing element can be considered as being of the two-dimensional type just discussed, but with this modification: that the air pressures and forces are determined from the motion of each wing element relative to the air in its vicinity, rather than from element's absolute motion. The air around the wing element has an average motion caused by the concentration of the lift over a finite span. Its downward component diminishes the angle of attack. Further, the resultant air force is orientated with respect to this relative motion, and hence possesses a component parallel to the absolute motion of the wing. This is the author's induced drag.

The air motion caused by the lateral lift concentration is again treated as a two-dimensional flow around a straight line, the line representing the entire span this time. The plane of the flow is at right angles to the direction of flight. The flow is gradually built up as the wing passes through the plane, hence, it is not steady as the wing section flow. The forces are, therefore, computed from the rate of change of the potential, not from the pressure. These are principles employed with the airship theory in Chapter II. The reader of the preceding chapters is thus already familiar with the mathematical and physical principles of the wing theory.

We discuss first the important case of equal downward velocity at all points of the span, and proceed then to variable downwash.

37. Physical Aspect

The last chapter does not give correct information on the aerodynamic wing forces, since the flow in vertical longitudinal planes was supposed to be two-dimensional. The vertical layers of air parallel to the motion were supposed to remain plane and parallel and only the distortion of the two other planes at right angles to it was investigated.

This is a very incomplete and arbitrary proceeding, for the vertical longitudinal layers do not remain plane, any more than any other layers remain plane. It is therefore necessary to complete the investigation and to assume now another set of layers, parallel to the lift, to remain plane, thus studying the distortion of the vertical longitudinal layers of air. Accordingly, we will assume that all vertical layers of air at right angles to the motion remain plane and parallel, so that the air moves at right angles to the direction of flight only. Hence, we have now to consider two-dimensional transverse vertical flows. This consideration, it will appear, gives sufficient information on the motion of the air at large, whereas the preceding investigation gives information on the conditions of flow in the vicinity of the wing. Both the longitudinal two-dimensional flow studied before and the two-dimensional flow to be studied presently, possess vertical components of velocity. Both flows, and more particularly these vertical components, are to be superposed, and thus the final aerodynamic pressures and resultant forces can be determined.

The transverse vertical layer of air is at rest originally. The wings, first approaching it, then passing through it and at last leaving it behind them, gradually build up and leave behind them a wake, described by a two-dimensional flow in each layer. The distribution of impulse creating this flow is identical with the distribution of the lift over the longitudinal projection of the wings. It is immaterial for the final effect whether all portions of the wings at every moment have transferred the same fraction of the momentum to a particular layer or not. The

final effect, and hence the average effect, is the same as if they have done it. They actually have if all wings are arranged in one transverse plane—that is, if the airplane is not staggered. It may be assumed at present that at each moment each layer has received the same fraction of the impulse from every portion of the wings, and it follows then that the shape of the configuration of the two-dimensional flow is always the same and that it is built up gradually by increasing its magnitude while not changing its shape, beginning with the magnitude zero at a great distance in front of the wing and having obtained its final magnitude at a great distance behind the wings.

The potential of the final two-dimensional flow long after the wings have passed through the layer is easy to find, for the impulsive pressure creating it is known along the longitudinal projection of the wings. It is identical with the distribution of the lift over this projection, acting as long as the airplane stays in the layer. This is the unit of time, if the thickness of the layer is numerically equal to the velocity of flight. Hence, the potential difference along the longitudinal projection of the wings is equal to the density of the lift along this projection divided by the product of the density of air and the velocity of flight, since the velocity potential is equal to the impulse of the pressure creating the flow, divided by the density. In general, the longitudinal projections of the wings can be considered as lines. The density of lift per unit length of these lines is then equal to the difference of pressure on both sides, and hence the density of the lift is proportional to the difference of the potential on both sides. This statement determines completely the final two-dimensional flow in the transverse vertical layer, and nothing remains unknown if the distribution of the lift over the wings is given. The actual determination of the flow is then a purely mathematical process.

For the present purpose, however, not the final transverse flow but the vertical flow at the moment of the passage of the wings is of interest. It is this flow that is to be superposed on

the longitudinal flow in order to determine the actual air forces. It has already been mentioned that this flow can be supposed to differ from the final flow in magnitude only. It remains, therefore, only to find the ratio of momentum already transferred while the wing passes through the layer, to the momentum finally to be imparted.

The fraction ½ is more plausible than any other fraction. The effect of the wing on the layer is the same at equal distances from the layer, whether in front or back of it and this would involve the factor ½. It is not necessary, however, to have recourse to a mere assumption in this question, however plausible it may be. It can be proved that the assumption of ½ is the only one which does not lead to a contradiction with the general principles of mechanics. We proceed at once to demonstrate this fact.

If the transverse flow in the plane of the wings is found, only the vertical component downward, u', called the induced downwash, is used for the computation. This downwash can be positive or negative, but in general is positive. Such downwash in the neighborhood of a portion of wing changes the motion of the air surrounding the wing portion relative to it. The induced downwash is always small when compared with the velocity of flight. Hence, its superposition on the velocity of flight at right angles to it does not materially change the magnitude of the relative motion between the wing and the air in its vicinity. It changes, however, the direction of this relative velocity, which is no longer parallel to the path of the wing, but inclined toward the path by the angle whose tangent is u'/V. This has two important consequences.

The flow produced and hence the air force, no longer are in keeping with the angle of attack between the wing and the path of flight but with the angle given by the motion of the wing relative to the surrounding portion of the air. In most cases the angle of attack is decreased and now the effective angle of attack, smaller than the geometric angle of attack between path

and wing, determines the flow and the air forces. Hence, the lift is in general smaller than would be expected from the geometric angle of attack. The angle of attack in the preceding chapter on the wing section is not identical with the geometric angle between the chord and the direction of flight but with the effective angle of attack, smaller in general, as there is an induced downwash motion in the vicinity of the wing. The difference is the induced angle of attack.

$$\alpha_i = u'/V \tag{1}$$

That is not all. The lift is not only decreased but its direction is changed, too. It is no longer at right angles to the path of

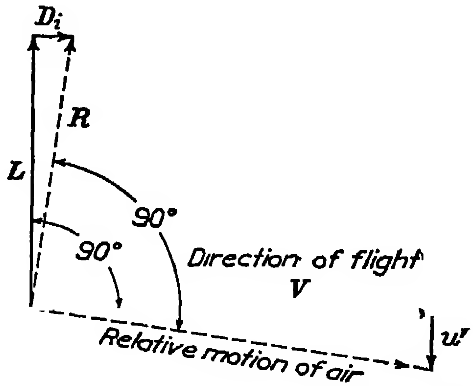


Figure 18. Diagram Showing the Creation of the Induced Drag

flight, but to the relative motion between wing and adjacent portion of air. It is turned backward through an angle equal to the induced angle of attack. The air force has now a component in the direction of the motion. The wing experiences an "induced" drag, in addition to the drag caused by the viscosity of the air, not discussed now, and the induced drag is often much larger than the viscous drag. The density of the

induced drag is $dL\ u'/V$, where dL is the density of lift, as can be directly seen from Figure 18.

$$dD_i = dL\ u'/V \tag{2}$$

The existence of a drag could have been anticipated, as there must be a source of energy for the creation of the transverse flow under consideration. The final kinetic energy of this flow in a layer of thickness V is equal to the integral of the energy imparted by each element of lift dL at the average velocity $u/2$

$$= \int \tfrac{1}{2}\, u dL$$

and this energy is to be delivered by the wing per unit of time, as during this unit of time another layer has been put into motion in the way discussed. On the other hand, the energy delivered by the wings is the integral of the drag multiplied by the velocity, that is, $\int u'/V \cdot V dL$. From which follows immediately $u' = \frac{1}{2}u$, and it is thus confirmed that the transverse flow is only half formed when the wings are passing through the vertical layer

38. Minimum Induced Drag

The problem is thus solved in general if the shape of the wings and the distribution of lift over the wings is known. Before passing to special wing arrangements and distributions of lift, in particular to the simple monoplane, there is one general problem to be discussed. The longitudinal projection of the wings being given, as well as the entire lift, the induced drag depends on the distribution of the lift over the projection. The drag is desired to be as small as possible. The question arises, What is the distribution of lift giving the smallest induced drag? The importance of this question is at once obvious.

The entire lift and the entire induced drag of the wings are found again as important characteristics of the final transverse flow, discussed in the last section. The resultant lift is equal to the resultant vertical momentum of this flow for the thickness of the layer equal to the velocity V, and the induced drag is

equal to the kinetic energy in the same layer divided by V. The problem is, therefore, to find such a two-dimensional flow produced by impulsive pressure over the longitudinal projection of the wings as possesses a given magnitude of the vertical momentum, and the kinetic energy of which is a minimum.

It is sufficient for elementary questions to consider only arrangements of wing symmetrical with respect to a vertical longitudinal plane, giving, moreover, horizontal lines in the longitudinal projections. The results are valid for all conditions. It is then easy to find the solution. The momentum of several flows superposed on each other is the sum of their single momenta. The flow is of the desired kind if the superposition of any other flow with the resultant vertical momentum zero increases the kinetic energy of the flow.

The velocity of the superposed flow can be assumed to be small, for instance, so that its own kinetic energy, containing the square of the velocity, can be neglected. The impulsive pressure along the projection of the wings necessary to create the superposed flow acts along a path determined by the magnitude of the downwash at the same points. The increase of kinetic energy is:

$$\tfrac{1}{2}\rho \int u' \Phi dx \qquad \text{(Eq. 14, Sec. 8)}$$

where $\int \Phi dx = 0$.

It is readily seen that the first expression can be identically zero for any distribution of the potential Φ restricted by the second condition only if the downwash u is constant over the entire projection of the wings. Only then a transfer of a portion of lift from one point to another with smaller downwash is impossible, whereas this proceeding in all other cases would lead to a diminution of the induced drag. It is thus demonstrated:

The induced drag is a minimum, if the transverse two-dimensional flow has a constant vertical component of velocity along the entire projection of the wings.

For wings without stagger it follows then that the induced angle of attack u'/V is constant over all wings.

The magnitude of the minimum induced drag of a system of wings is easily found from the apparent mass ρK of their longitudinal projection in the two-dimensional transverse flow. For the vertical momentum equal to the lift is $uV\rho K = L$, where u is the constant downwash of the final flow. This gives

$$u = L/V\rho K$$

The induced drag is equal to the kinetic energy divided by V

$$D_i = u^2 \rho/2 \cdot K$$

It follows, therefore, that the minimum induced drag is

$$D_i = \frac{L^2}{4V^2 \rho/2 \cdot K} \tag{3}$$

and the constant or at least average induced angle of attack is

$$\alpha_i = \frac{u'}{V} = \frac{L}{4V^2 \rho/2 \cdot K} \tag{4}$$

K is a constant area determined by the longitudinal projection of the wings. It is the area of the air in the two-dimensional flow having a mass equal to the apparent mass of the projection of the wings.

The results (3) and (4) show that the minimum induced drag can be obtained from the consideration that the lift is produced by constantly accelerating a certain mass of air downward from the state of rest. The apparent mass accelerated downward is at best equal to the apparent mass of the longitudinal projection of the airplane in a layer of air passed by the airplane in the unit of time.

In practical applications the actual induced drag can be supposed to be equal to the minimum induced drag, and the average induced angle of attack equal to (4). It is, of course, slightly different, but the difference is not great, as can be expected since no function changes its value much in the neighborhood of its minimum.

We proceed now to the application of the general theory of induction to the case of the monoplane without dihedral angle, giving in longitudinal projection a straight line of the length b. Consider first the distribution of lift for the minimum induced drag. It is characterized by the transverse potential flow with constant vertical velocity component along this straight line. This flow has repeatedly occurred in the earlier chapters. For the length 2 of the line, it is the transverse flow given by equation (1) of Section 24 or by equation (3) of Section 24 when $n = 1$. The potential function for the length b is:

$$F = A_1 i \left[\frac{2z}{b} \pm \sqrt{\left(\frac{2z}{b}\right)^2 - 1}\right] \tag{5}$$

giving the constant vertical velocity component along the line

$$u = A_1\, 2/b$$

The density of the lift per unit length of the span is equal to the potential difference of the final flow on both sides of the line, multiplied by $V\rho$.

$$\frac{dL}{dx} = 2V\rho A_1 \sqrt{1 - \left(\frac{2z}{b}\right)^2} = 2A_1 V\rho \sin\delta = \frac{4L \sin\delta}{b\pi} \tag{6}$$

where $\cos\delta = 2x/b$. This follows from

$$L = \int_{-b/2}^{+b/2} \frac{dL}{dx}\, dx = \int_{-b/2}^{+b/2} 2V\rho A_1 \sqrt{1 - \left(\frac{2z}{b}\right)^2}\, dz = V\rho A_1 \pi \frac{b}{2}$$

Hence
$$A_1 = \frac{2L}{V\rho\pi b}$$

Plotted against the span, the density of lift per unit length of the span is represented by half an ellipse, the multiple of $\sin\delta$ being plotted against $\cos\delta$. The lift, therefore, is said to be elliptically distributed.

The apparent mass of the line with the length b is equal to

$$\rho K = b^2\, \pi/4 \cdot \rho$$

Hence, with this distribution of lift, the minimum induced drag is, according to equation (3)

$$D_i = \frac{L^2}{b^2 V^2 \rho/2 \cdot \pi} \tag{7}$$

and the constant induced angle of attack according to equation (4) is

$$\alpha_i = \frac{L}{b^2 V^2 \rho/2 \cdot \pi} \tag{8}$$

The density of lift (6) per unit length of the span together with the chord c, different in general along the span, and with equation (22), Chapter IV, determines the effective angle of attack at each point, including the apparent angle of attack of the section. Substituting from equation (6) gives the effective angle of attack:

$$\alpha_e = \frac{\text{density of lift}}{2\pi \text{ chord } V^2 \rho/2} = \frac{dL/dx}{2\pi c V^2 \rho/2} = \frac{2L \sin\delta}{b\pi^2 c V^2 \rho/2} \tag{9}$$

The geometric angle of attack is greater by the constant induced angle of attack, and hence

$$\alpha_g = \frac{2L \sin\delta}{b\pi^2 c V^2 \rho/2} + \frac{L}{b^2 \pi V^2 \rho/2} = \alpha_e \left(1 + \frac{\pi c}{2b \sin\delta}\right) \tag{10}$$

Equation (7) indicates the importance of a sufficiently large span in order to obtain a small induced drag.

Any distribution of lift dL/dx over the span other than the elliptical distribution is less simple to investigate, as then the induced downwash is variable. The distribution of lift gives directly the distribution of the potential difference along the two-dimensional wing projection.

$$\Delta\Phi = \frac{dL/dx}{V\rho} \tag{11}$$

The transverse two-dimensional flow can now be obtained by superposition of types of flow given by equation (3) of Section 24 with $z = 2x/b$, as now the length of the line is not 2 but b.

The condition is that the superposition of such flows gives the required potential difference; viz.,

$$\tfrac{1}{2}\Delta\Phi = \frac{dL/dx}{2V\rho} = A_1 \sin\delta + A_2 \sin 2\delta + \ldots + A_n \sin n\delta + \ldots \quad (12)$$

Hence, the distribution of the density of lift, divided by $2V\rho$ is to be expanded into a Fourier's series. The induced angle of attack results then, according to equation (12) of Section 24,

$$\alpha_i = \frac{u'}{V} = \frac{1}{bV \sin\delta}(A_1 \sin\delta + 2A_2 \sin 2\delta + \ldots + nA_n \sin n\delta + \ldots) \quad (13)$$

and the entire induced drag, being the integral of the product of the induced angle of attack and the density of lift with respect to the element of the span is:

$$D_i = \int_{-b/2}^{+b/2} \frac{dL}{dx}\alpha_i dx = \rho\frac{\pi}{2}(A_1^2 + 2A_2^2 + \ldots + nA_n^2 + \ldots) \quad (14)$$

As mentioned before, the induced drag for any reasonable distribution of lift agrees practically with the minimum induced drag, given by equation (7). A_1 is the main coefficient and all other A's are then small when compared with it. It is, therefore, exact enough for practical problems to apply equation (7) indiscriminately, whether the distribution of lift is exactly elliptical or only within a certain approximation. In the same way, equation (8) for the constant induced angle of attack can be used generally for the average induced angle of attack.

39. The Elliptic Wing

In practice the reverse problem is more often met with. Not the distribution of the lift but the shape of the wing is known; viz., the magnitude of its chord and the angle of attack at each point. The air forces are to be determined.

The solution of this problem in full is usually barred by great mathematical difficulties. There is, however, one particular

plan view of the wing which can be treated comparatively simply and which gives very interesting results. That is the elliptical wing; that is, a wing with such a plan view that the chord plotted against the span is represented by half an ellipse.

A possible way to investigate a wing with a given plan view would be to look for particular distributions of the angle of attack such that the solution for them can be found. It would be particularly easy to use such special solutions for the solution of the general problem, if it were possible to determine those particular special solutions, for which the induced angle of attack is proportional to the effective angle of attack and hence to the geometric angle of attack also. These functions found, the distribution of the angle of attack arbitrarily given must be expanded as a series of such functions, that is, as a sum of them. It is then easy to find from this series the induced angle of attack or the effective angle of attack, as this can be done for each term separately by the mere multiplication with a certain constant. Hence, a new series for the effective angle of attack is readily obtained from the series of geometric angles of attack.

That sounds simple, but it is extremely difficult to find such distributions of the angle of attack of a given plan view. It suggests itself, therefore, to try the other way and to begin with simple distributions of the angle of attack and to try to find a plan view which can be conveniently investigated by means of them. The only distributions discussed in this book are those represented by the flows equation (3), Section 24. It suggests itself to begin by considering the induced angle of attack and effective angle of attack. For one special term, as follows from equations (9), (12), and (13), the effective angle of attack is:

$$\alpha_e = \frac{2A_n \sin n\delta}{\pi c V} \tag{15}$$

and the induced angle of attack is:

$$\alpha_i = \frac{nA_n \sin n\delta}{bV \sin\delta} \tag{16}$$

It is at once seen that these two angles become proportional to each other if the chord c becomes proportional to $\sin\delta$. For circles and ellipses

$$c = \frac{S}{b\,\pi/4}\sin\delta \tag{17}$$

Hence, substituting this in (15)

$$\alpha_e = \frac{A_n \sin n\delta}{Vb \sin\delta\; 2S/b^2}$$

where S denotes the entire area of the wing. The ratio of the two angles of attack becomes then

$$\alpha_i/\alpha_e = 2nS/b^2 \tag{18}$$

This holds primarily for elliptical plan views, but can be applied to others also as an approximation.

With an elliptical wing, all equations found for any monoplane become particularly simple. Equation (10) can be written

$$\alpha_g = \alpha_e\left(1 + \frac{2S}{b^2}\right) \tag{19}$$

The geometric angle of attack follows from equations (15) and (16) for the effective and induced angles of attack, valid for a special n, by taking the sum of all such expressions. The condition for the coefficient A, if the geometric angle of attack α_g is given, is therefore,

$$\frac{2VS}{b}\alpha_g \sin\delta = A_1 \sin\delta\left(1 + \frac{2S}{b^2}\right) + A_2 \sin 2\delta\left(1 + \frac{4S}{b^2}\right) + \cdots$$
$$+ A_n \sin n\delta\left(1 + \frac{2nS}{b^2}\right) + \cdots \tag{20}$$

That is, $2V\alpha_g \sin\delta \cdot S/b^2$ is to be expanded into a Fourier's series

$$2V\alpha_g \sin\delta \cdot S/b^2 = B_1 \sin\delta + B_2 \sin 2\delta + \cdots + B_n \sin n\delta + \cdots \tag{21}$$

and the coefficients A_n are then

$$A_n = \frac{B_n}{1 + 2nS/b^2} \tag{22}$$

The distribution of the lift follows then from equation (11)

$$dL/dx = 2V\rho(A_1 \sin\delta + A_2 \sin 2\delta + \ldots + A_n \sin n\delta + \ldots) \quad (23)$$

and the distribution of the induced angle of attack is given by equation (13). The entire induced drag is given by equation (14).

The entire lift is

$$\int_{-b/2}^{+b/2} \frac{dL}{dx} dx$$

and only the first term of series (23) contributes to it in view of formula (24), since $dx = -\sin\delta d\delta$.

$$\int_0^{\pi} \sin n\delta \sin m\delta d\delta = 0 \quad \text{if } n \neq m \quad (24)$$

Hence, the entire lift is

$$L = 2V\rho A_1 \int_{-b/2}^{+b/2} \sin\delta dx$$

or transformed by introducing α_g, and using (21) and (22),

$$L = V^2\frac{\rho}{2} \frac{1}{1 + 2S/b^2} 2 \int_{-b/2}^{+b/2} \alpha_g c dx \quad (25)$$

That is to say, the entire lift of an elliptic wing can be obtained by supposing the effective angle of attack equal to the geometric angle of attack divided by $(1 + 2S/b^2)$. Otherwise expressed, the aerodynamic induction reduces the entire lift of the elliptical wing in the ratio

$$\frac{1}{1 + 2S/b^2}$$

however the wing may be twisted.

The usefulness of this result must not be overestimated. The distribution of the lift itself is by no means obtained by a mere diminution of the angle of attack in a constant ratio; only

the entire lift can be computed that way. Nor does this theorem hold true for any other plan view but the elliptical. It may, however, be applied to the ordinary wing shapes which are approximately elliptical so as to gain approximate results.

40. Induction Factor for the Rolling Moment

The rolling moment of the lift can be found by means of quite an analogous theorem. Only the second term in (23) gives a contribution to the rolling moment in view of formula (24). This is, therefore, as a consequence of formulas (23), (22), (21), and (17),

$$M = \frac{V^2\,\rho/2 \cdot 2\pi}{1 + 4S/b^2} \int_{-b/2}^{+b/2} c\alpha x dx \tag{26}$$

Hence, the entire rolling moment is obtained by taking as the effective angle of attack, the geometric angle of attack times

$$\frac{1}{1 + 4S/b^2}$$

The induction decreases the rolling moment in the same ratio. This is of interest for the computation of the effect of a displacement of ailerons.

It will be noticed that this factor of decrease is a different one for the entire lift and for the entire rolling moment. The induced angle of attack is not proportional to the geometric angle of attack except when all factors but A_1 are zero. This is true in the main case $n = 1$, where the constant angle of attack is decreased by a constant induced angle of attack, as we have seen before.

Equation (26) may also be applied as approximation to shapes differing from elliptical plan forms. The error involved in this proceeding is probably greater in general than for the determination of the entire lift, as the rolling moment is more influenced by the ends of the wing and there the deviation from the elliptical shape will be particularly pronounced.

41. Yawing and Rolling Moment

We consider now a displacement of the ailerons during flight. The lift is increased on one end of the airfoil and it is decreased on the other end. But the drag undergoes changes, too, and is usually increased or decreased on the same ends where the lift is increased or decreased. Thus, in addition to the rolling moment, which is desired, a yawing moment is set up by the displacement of the ailerons, and, according to what is said, it is an undesirable one acting against the rudder. For instance, when turning from a straight flight path to the left, the right side of the airplane must be banked up and must be yawed ahead of the left side. But the yawing moment, just mentioned, tends to yaw the left side ahead of the right side. It is, therefore, desirable to have this yawing moment, as small as possible. The following discussion deals with the magnitude of this yawing moment and leads to a numerical relation between it and the quantities which chiefly govern its size. This relation is of interest and use in all cases where the magnitude of the yawing moment occurs; that is, with all questions of stability and controllability.

The displacement of an aileron is equivalent to the change of the wing section, of which it forms a part. This change of section will generally cause a change of the friction drag. The magnitude of this change depends upon many factors, and it is difficult to make a general statement concerning it. It can be said, however, that it is not so very large, except near the angle of attack of maximum lift, and that the changes are not necessarily of opposite sign on both wing ends. The changes of the induced drag will be much larger in most cases, and these changes are of opposite sign, giving rise to a yawing moment, directed as stated above. This induced yawing moment, forming probably the main part of the entire yawing moment encountered by the wings, lends itself readily to an analytical investigation. We will proceed, therefore, to compute the induced yawing moment of the wing, making assumptions which greatly simplify

the mathematical treatment without essentially specializing the problem. On the contrary, the solution will be a good approximation for all practical cases.

We consider a single airfoil moving in an ideal fluid and assume the lift per unit length of the span to vary in proportion to the ordinates of an ellipse, with the span as a main axis. Let b denote the length of the span and introduce the angle δ by means of

$$b/2 \cdot \cos\delta = x \tag{27}$$

where x denotes the distance of a wing element from the middle of the wing. Then the elliptical lift distribution can be expressed by

$$L' = dL/dx = 2V\rho C \sin\delta \tag{28}$$

where V denotes the velocity of flight,

ρ denotes the density of the air, and

C is a constant of the dimension of a velocity potential.

The entire lift is then

$$L = \int_{-b/2}^{+b/2} L'dx = 2CV\rho\frac{b}{2}\int_0^{\pi}\sin^2\delta d\delta$$

$$L = CV\,\rho/2 \cdot b\pi$$

whence

$$C = \frac{1}{\pi}\frac{L}{bV\,\rho/2} \tag{29}$$

The distribution of the lift produced by a displacement of the ailerons is assumed to be an odd function of x. Then it can be expanded in a Fourier's series with even multiples of δ only, giving the entire lift distribution,

$$L' = 2V\rho(C\sin\delta + A_2\sin 2\delta + A_4\sin 4\delta + \ldots) \tag{30}$$

where the A's are any constants of the same kind as C.

The entire rolling moment is then

$$M_r = \int_{-b/2}^{+b/2} L'xdx$$

$$2V\rho\left(\frac{b}{2}\right)^2\int_0^{\pi}(C\sin\delta + A_2\sin 2\delta + \ldots)\,\frac{\sin 2\delta d\delta}{2}$$

$$M_r = A_2V\rho b^2\,\pi/8$$

whence

$$A_2 = \frac{4}{\pi}\frac{M_r}{b^2 V\,\rho/2} \tag{31}$$

The induced angle of attack can be written by using equation (5), Section 24,

$$\alpha_i = \frac{1}{bV\,\sin\delta}\,(C\,\sin\delta + 2A_2\,\sin 2\delta + 4A_4\,\sin 4\delta + \ldots) \tag{32}$$

The distribution of the induced drag follows then from

$$dD/dx = D_i' = \alpha_i L'$$

$$D_i' = \rho\frac{2}{b}\frac{1}{\sin\delta}(C\,\sin\delta + A_2\,\sin 2\delta + A_4\,\sin 4\delta\,\ldots)$$
$$(C\,\sin\delta + 2A_2\,\sin 2\delta + 4A_4\,\sin 4\delta\,\ldots) \tag{33}$$

The resulting induced yawing moment is

$$M_y = \int_{-b/2}^{+b/2} D_i' x\,dx = \frac{b^2}{4}\int_0^{\pi} D_i'\,\frac{\sin 2\delta}{2}\,d\delta \tag{34}$$

or substituting (33)

$$M_y = \frac{\rho b}{2}\int_0^{\pi}\cos\delta\,(C\,\sin\delta + A_2\,\sin 2\delta + A_4\,\sin 4\delta\,\ldots)$$
$$(C\,\sin\delta + 2A_2\,\sin 2\delta + 4A_4\,\sin 4\delta\,\ldots)$$

This integral can be split into the three following ones:

$$M_y =$$

$$\text{(I)}\quad \frac{\rho b}{2}\int_0^{\pi}\cos\delta C_2\,\sin\delta d\delta$$

$$\text{(II)}\quad + \frac{\rho b}{2}\int_0^{\pi}\cos\delta C\,\sin\delta\,(3A_2\,\sin 2\delta + 5A_4\,\sin 4\delta + \ldots)\,d\delta$$

$$\text{(III)}\quad + \frac{\rho b}{2}\int_0^{\pi}\cos\delta(A_2\,\sin 2\delta + A_4\,\sin 4\delta + \ldots)$$
$$(2A_2\,\sin 2\delta + 4A_4\,\sin 4\delta + \ldots)\,d\delta$$

The integrands I and III are the products of a symmetric func-

tion and cos δ, and hence the integrals are zero. The integral II can be written

$$M_y = \frac{\rho b}{2}\int_0^\pi \frac{C}{2}\sin 2\delta(3A_2\sin 2\delta + 5A_4\sin 4\delta + \ . \ . \ .)\,d\delta$$

which gives

$$M_y = \rho/2 \cdot bCA_2 3/4 \cdot \pi$$

Substituting (29) and (31) gives the final result

$$M_y = M_r\frac{3}{\pi}\frac{L}{b^2V^2\,\rho/2}$$

or otherwise written

$$\frac{M_y}{M_r} = \frac{3}{\pi}C_L\frac{S}{b^2}$$

or; $\dfrac{\text{Induced Yawing Moment}}{\text{Rolling Moment}}$ is about $\dfrac{\text{Lift Coefficient}}{\text{Aspect Ratio}}$

42. The Biplane

All results obtained are also valid for biplanes and multiplanes, if care is taken to substitute the apparent mass of the front view of the multiplane for that of the monoplane. The computation of the apparent mass of a pair of parallel lines, representing a biplane is comparatively simple. There is moreover an approximate method for obtaining the area of apparent mass of a biplane, exact enough for the needs of practice.

It is obtained from the consideration that the flow on top and under the pair of lines is in substantial agreement with that under and over the single line, and hence, the area of apparent mass for this portion of the flow is likewise equal to the area of the circle over the span as diameter. In addition, there is the fluid enclosed between the wings. It moves almost like a solid together with the wings, and hence its area of apparent mass is equal to the area enclosed between the wings. The entire area is, therefore, $b^2\pi/4 + bh$ where b denotes the span and h the gap. The comparison with the area exactly computed shows a very good agreement with this approximate formula.

The induced drag and angle of attack can now be computed by the use of the monoplane formula, if a fictitious span b' is substituted for the actual span. This effective span is larger than the actual one, and is so chosen that the area of the circle over it as diameter is equal to the area of apparent mass of the biplane. Hence, $b' = \sqrt{b^2 + 4bh/\pi}$.

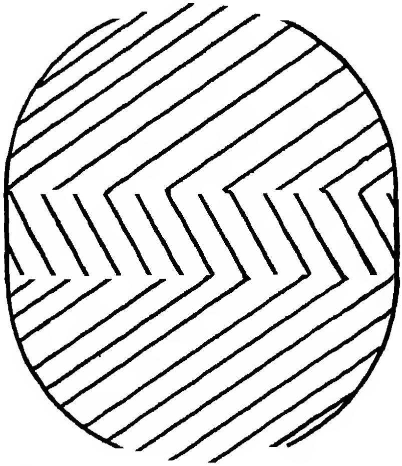


Figure 19. Area of Apparent Mass of a Pair of Lines

It appears then, that the biplane has a smaller induced angle of attack and drag than a monoplane for the same lift. This must not be misunderstood, the monoplane is supposed to have the same lift as both biplane wings together. In most cases monoplanes have smaller lift, and then their induced drag may well be smaller than that of biplanes.

43. Synopsis

The induced angle of attack of a single wing is:

$$\alpha_i = \frac{L}{\pi b^2 V^2 \rho/2} \text{ in radians}$$

The effective angle of attack is the difference of the geometric or actual angle of attack and this induced angle of attack.

The induced drag of a single wing is:

$$D_i = \frac{L^2}{\pi b^2 V^2 \rho/2}$$

The downwash diminishes the lift of a single wing in the ratio

$$1 : (1 + 2S/b^2)$$

compared to the lift computed under the same conditions from the two-dimensional flow. The downwash diminishes the rolling moment in the ratio

$$1 : (1 + 4S/b^2)$$

For biplane cellules, replace b in these formulas by b', where

$$b'^2 = b^2 + 4bh/\pi$$

The induced yawing moment is equal to

$$\frac{\text{Rolling Moment} \times \text{Lift} \times b^2}{S^2\ V^2\ \rho/2},$$

where S denotes wing area, b span, h gap, V velocity of flight, ρ density of air.

44. Problems and Suggestions

1. The span of a single rectangular wing is 30 ft., the lift 1,200 lb., the velocity 100 ft./sec. and the density of the air 1/420 lb. ft^4/sec^2. How large is the induced angle of attack?

2. How large is the induced drag in Problem 1?

3. Two single wings as in Problem 1 are joined together to a biplane cellule with a gap of 5 ft., the lift per wing is the same and likewise the velocity of flight and the density of the air. Compute the induced drag and the induced angle of attack.

4. The area of a single wing is 150 sq. ft., the span 30 ft., the velocity 100 ft./sec. and the density of the air 1/420 lb. sec^2/ft^4. By how much is the lift increased if the angle of attack is increased by 2°?

5. The span of the wing in Problem 1 is diminished by 5 ft.

The chord remains 5 ft. By how much has the angle of attack to be increased in order to carry again the same weight at the same speed?

6. The wing remains the same as in Problem 1, but the density of the air is diminished by 25%, the airplane having climbed to a corresponding altitude. By how much must the angle of attack be increased to carry the same load at the same speed?

7. Compute the induced angle of attack and the induced drag for the last problem.

8. The span of a rectangular plane stabilizer and elevator is 10 ft., the chord of the stabilizer is 3 ft. and that of the elevator 1 ft. The velocity of flight is 100 ft./sec., the air density 1/420 lb. sec^2/ft^4 the angle of attack of the stabilizer is 1° and the displacement of the elevator 3°. How large is the stabilizer lift, if the influence of the airplane wings be neglected?

9. This elevator is behind the wing in Problem 1 and the downwash produced near the elevator by the wing is twice that at the wing. How large is the lift of the elevator, if the angle of attack of the stabilizer is 4°, all other quantities being the same as in Problem 8?

10. The lift of the elevator and stabilizer in the last problem is to be increased by 10% by leaving the area and the ratio of the elevator chord to the stabilizer chord, and changing only the ratio of the span to the chord. How large must the span be?

11. Two glider wings with the aspect ratios 14.5 and 17 have equal chord and equal ailerons. The ailerons are displaced by an equal angle while the speed and the density of air are equal also. What is the ratio of the produced rolling moments, if the spans of the ailerons are one-fifth of the airplane span of the larger wing?

12. A wing with the span 36 ft., the area 250 sq. ft., the velocity 120 ft./sec., flying in air of the density 0.0020 lb. sec^2/ft^4 has the lift 1,400 lb. at an angle of 3°. At which angle of attack is the lift zero?

CHAPTER VI

PROPELLER THEORY

45. Outlook

The propeller theory is quite similar to the wing theory. The blade elements are regarded as wing elements working in air having an average motion of its own. This average motion, the slipstream flow, is now an axial jet.

The computation of the air forces is more elaborate than with wings, since all blade elements are different and are working under different conditions. It is, however, possible to determine two characteristic quantities for each propeller representing the average of all elements, and to compute the air forces under all conditions of flight from these two quantities.

46. Slipstream Theory

The propeller action is computed by regarding the blade elements as small wing elements working in the slipstream. The flow around each blade element is again described by a two-dimensional wing-section flow and it is again determined by the motion of the blade element relative to the surrounding air, rather than by its absolute motion.

The slipstream theory[1] treats with the average motion of the air. This velocity distribution affects the magnitude of the thrust appreciably and causes the efficiency to be smaller than 1, even in a perfect fluid.

The slipstream flow is much simpler than the downwash flow of a wing. The air flows from all sides into the front of the propeller circle and leaves it in a circular jet. This jet reaches its highest velocity v' far behind the propeller. Its velocity at

[1] See reference 2, page 175.

the propeller may be denoted by v. It may be assumed at first to be equal at all points of the propeller. It can be shown that $v = v'/2$. The mass of the air passing through the propeller

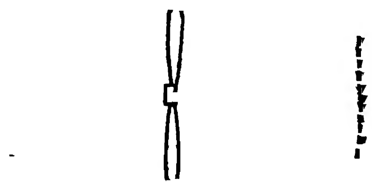


Figure 20. Propeller Slipstream

disc is equal to $(V + v)\rho\ D^2\ \pi/4$. The momentum imparted to the air per unit of time is equal to the thrust T, equal to its increase of momentum

$$T = (V + v)\rho D^2\ \pi/4 \cdot v' \tag{1}$$

The energy imparted to the air per unit of time is Tv, equal to its increase of kinetic energy

$$Tv = (V + v)\rho D^2\ \pi/4 \cdot v'^2/2$$

The combination of these two equations gives $v' = 2v$ and

$$v/V = \tfrac{1}{2}\left(\sqrt{1 + \frac{8T}{V^2D^2\rho\pi}} - 1\right) \tag{2}$$

Since the thrust must be exerted along the velocity $(V + v)$ but performs useful work along the velocity V only, the theoretical efficiency of the propeller is

$$\eta = \frac{V}{V + v} = \frac{2}{1 + \sqrt{1 + 8T/V^2D^2\rho\pi}} \tag{3}$$

This gives all information necessary for the application of the wing theory to the blade sections. The rotation of the slipstream can be neglected.

With a propeller in the test stand, or a helicopter, the velocity V becomes zero. Equation (1) assumes then the shape

$$T = 2v^2\rho D^2\,\pi/4$$

Hence the slip velocity becomes

$$v = \sqrt{2T/\rho D^2\pi}$$

and the power absorbed theoretically

$$P = \sqrt{2T^3/\rho D^2\pi}$$

47. Effective Pitch and Blade Size

The application of the wing section theory to propellers is more laborious than that to wings, as all blade sections are different and working under different conditions. This computation can be simplified by the introduction of the effective pitch and the effective blade size.

The effective pitch of a blade element is the pitch of its spiral path of zero thrust. The effective pitch of the whole propeller is the pitch of the spiral path of zero resultant thrust of the propeller. It is found by summing up the effects of all blade elements.

The second important characteristic of a propeller results from a computation of the thrust. Let the geometric angle of attack of a blade element be α. Then $\cot \alpha = U'/(V + v)$ where U' denotes the tangential velocity component of the element, V the velocity of flight and v the slipstream velocity at the element. We proceed by keeping V constant and varying U' slightly by the increment dU' from the tangential velocity U'_0 of zero thrust. This requires $U'_0/V = 2r\pi/p$, where p is the effective pitch of the element and r its distance from the axis.

The cotangent of the angle is now

$$\cot(\alpha_0 + d\alpha) = \frac{U'_0 + dU'}{V + dv}$$

Hence

$$d(\cot\alpha) = \frac{dU'}{V} - \frac{U'_0 dv}{V^2}$$

On the other hand $d(\cot\alpha) = -\,d\alpha(1 + \cot^2\alpha)$,

hence
$$d\alpha = \frac{U'_0dv - VdU'}{U'^2_0 + V^2}$$

This is the negative of the effective angle of attack. Hence the thrust is

$$dT = 2\pi nc\, drd\alpha(V^2 + U'^2_0)\,\rho/2$$

where c denotes the projection of the blade width on the propeller disc, and n the number of blades. Introducing U, the tangential velocity component of the propeller tip by means of $U = U'D/2r$ where D the propeller diameter, gives finally

$$dT = 2\pi nc\rho(VdU - U_0dv)\frac{rdr}{D}$$

This is equal to the thrust computed from the slip velocity

$$dT = 4\pi rVdvdr\rho$$

giving by elimination of dT

$$dv = \frac{n/2 \cdot c/D \cdot dU}{1 + n/2 \cdot c/D \cdot D\pi/p} \tag{4}$$

Assuming now a linear relation between the tip velocity, the slip velocity and the velocity of flight, an assumption likely in itself and justified by experience, this linear relation is necessarily

$$v = \frac{n/2 \cdot c/D}{1 + n/2 \cdot c/D \cdot D\pi/p}(U - V \cdot D\pi/p) \tag{5}$$

This is the only linear relation in keeping with (4), and giving zero thrust at the pitch p of the spiral path of the element.

In many cases the blade width can be considered constant and equal to the average blade width. Then (5) can be written

$$v = \frac{S/D^2}{1 + S/D^2 \cdot D\pi/p}(U - V \cdot D\pi/p) \tag{6}$$

where S denotes the blade area $ncD/2$. This is the main formula for an abbreviated propeller computation.

A more exact computation considering the variation of the blade width, can be made by computing the momentum of the slipstream by summing up the momentum inside each ring zone. This can be simplified by using at least a constant pitch. This assumption leads to the same expression as (6) with

$$4/D^3 \int_0^{D/2} crdr$$

substituted for S/D^2. This integral is equal to the product of the projected area times the distance between its center of gravity and the axis.

Propellers with equal pitch ratio p/D and with equal blade size ratio

$$4/D^3 \int_0^{D/2} crdr$$

are equivalent within the scope of this theory, experiencing equal air forces under equal conditions.

48. Synopsis

The propeller dimensions are given:

1. Determine the average effective pitch, p.
2. The average slipstream velocity v at the propeller is

$$v = \frac{S/D^2}{1 + S/D^2 \cdot D\pi/p} (U - V \cdot D\pi/p)$$

The thrust is

$$T = 2D^2 \frac{\pi}{4} (V + v)v\rho$$

The theoretical efficiency is

$$\eta = \frac{V}{V + v} = \frac{2}{1 + \sqrt{1 + 8T/V^2D^2\rho\pi}}$$

The theoretical power is

$$N = \frac{TV}{1 - v/V}$$

The magnitude of the thrust is known or required. The average slip velocity is

$$v = \frac{V}{2}\left(\sqrt{1 + \frac{T}{D^2\pi/4 \cdot V^2\rho/2}} - 1\right)$$

49. Problems and Suggestions

1. The effective pitch of a propeller is 8 ft., and its speed 1,600 r.p.m. At which velocity of flight does the thrust become zero?

2. The velocity of flight for the same propeller is 140 ft./sec. At which r.p.m. is the thrust zero?

3. A propeller has a diameter of 10 ft., a pitch of 8 ft., and an average blade width of 11 in. The velocity of flight is 140 ft./sec., the speed 1,600 r.p.m. and the air density 1/420 lb. sec^2/ft^4. Compute the average slip velocity at the propeller.

4. Compute the thrust.

5. Compute the horsepower theoretically required.

6. Compute thrust and horsepower for the propeller in Problem 3 for an air density half that large.

7. The propeller in Problem 3 is cut shorter by half a foot at each end and the speed is increased to 1,800 r.p.m. How large is the thrust now?

8. Compute the r.p.m. for the propeller in Problem 3 necessary to create a thrust of 1,750 lb.

9. The airplane in Problem 3 is taxying with a velocity of 60 ft./sec., the propeller speed is 1,200 r.p.m. The elevator is in the slipstream. Compute the air velocity relative to the elevator, if the slipstream near the elevator is twice the slip velocity at the propeller.

10. A competitive project to the propeller in Problem 3 has a geared propeller with 3/5 of the speed and equal tip velocity. How much horsepower can be saved theoretically by gearing down?

CHAPTER VII

ADVANCED SUBJECTS

50. Outlook

A book on aerodynamics would be incomplete at present without at least some discussion on the subject of vortices. We consider vortices as unsuitable for the use of the designer, and prefer the methods given in the preceding chapters for obtaining the same results. Some remarks on vortices will prevent a misconception about their usefulness.

We attach a few remarks on sources. They are quite analogous to vortices, can be used in a similar way for the solution of a certain class of problems, and may some day receive more publicity than they have obtained at present.

The Joukowsky wing sections and the Karman vortices deserve a place in this book, not only on account of their publicity, but also on account of their intrinsic scientific value. They are not used generally in technical work.

The section on wind tunnel correction is so closely related to the wing theory that it is permissible to insert it in this book. Wind tunnel tests are constantly used by designers, reference to these corrections is frequently made; the theory of these corrections may, therefore, be welcome to the reader.

We close with a discussion of the best thrust distribution along the propeller blade. The friction forces are taken into account, nevertheless this discussion belongs in the theoretical part. It shows why the theory of the best lift distribution should not be generalized to propellers, as has been done by way of extending our lift theory.[1] It is not independent of the fric-

[1] See reference 1, page 175.

tion forces, as is the lift distribution, but is more influenced by the friction losses than by the slipstream losses.

51. The Vortices and Their Relation to the Lift

The wing theory is often called vortex theory because the first results were obtained by using vortices rather than the velocity potential. At present, a book on aerodynamics would seem incomplete without some reference to these vortices.

At the beginning of Section 1, we introduced the rotation of the fluid as twice the average angular velocity of a fluid particle. Vorticity is another name for this same thing, and was introduced into literature by Helmholtz in 1858. The vorticity has accordingly the components

$$v_z - w_y;\ w_x - u_z;\ u_y - v_x$$

Substitute these three expressions for u, v, and w in the equation of continuity (equation 12, Chapter I). We obtain

$$(v_{xz} - w_{xy}) + (w_{xy} - u_{yz}) + (u_{yz} - v_{xz})$$

and see at once that this is identically zero, as each term appears twice with opposite sign. This has been obtained without any restriction of the velocity distribution. We recognize, therefore, this relation to be a mathematical consequence of the definition of vorticity. However the velocity be distributed, its vorticity is always continuous like an incompressible fluid. Hence, we can picture a fictitious fluid flowing such that at each point its velocity is equal to the vorticity of our original flow. This fictitious fluid flows then always like a fluid of constant density.

The flow lines of the derived flow are properly called vortex lines. As always with an incompressible fluid, it is possible to divide the space into tubes, all bounded by vortex lines, so that an equal volume of the fictitious fluid passes through each cross-section during the same time. In particular, the tubes can be chosen so that the volume per unit of time is unity, and each tube can then be represented by its axis, which may be called a unit vortex line. This is much the same trend of thought as

used for the explanation of the magnetic force lines. No unit vortex line can end or begin amidst the fluid.

We proved in equation (7), Chapter I, that the rotation of a fluid particle cannot change unless there are external forces bringing about such change. Hence, vortex lines consist always of the same fluid particles. This physical property of a non-viscous fluid justifies the introduction of the vorticity. With a steady flow, then, the vortex lines are always coinciding with streamlines. This comparatively simple and easy theory is valuable for the theoretical investigation of flows that are not potential flows. Some theory is then required as the potential cannot be used for a flow without potential. We shall see later how the vortices are then used. Strange to say, however, during all these years of theoretical airfoil research the vortices have been used for the investigation of potential flows devoid of vorticity. It is this contradiction that makes the vortex theory of lift somewhat obscure and in our opinion unsuited for engineers.

We have not yet defined a vortex, for a vortex is not vorticity. A vortex is a bundle of unit vortices crowded densely together along some line, and its strength is measured by the number of unit vortices forming the vortex. A region approximated by such vortex may exist in a flow otherwise a potential flow. It consists then always of the same particles, as we have seen, and in steady flows coincides with a streamline.

The theory is now becoming more and more abstract. We imagine a vortex in a steady flow not being a streamline. This is only possible by the particles at the vortex being under the action of a suitable external force. In this way we arrive at the lift, which is the reaction to this external force. This explains in some way why a wing of infinite span is said to be equivalent to a vortex parallel to it, but the statement also implies that the flow caused by the wing is equal to a flow without any wing, but a vortex at its place. The strength of this vortex plays then the same part as the circulation, and indeed it is the

very same thing. Each vortex line has, of course, a circulation, otherwise it would not be incompatible with a potential flow. The circulation of the vortex is the sum of the circulations of all unit vortex lines of which it is composed.

We proceed now to explain how the vortex lines are used for the computation of the induced downwash. We first have to determine the distribution of vortices behind a wing and from these vortices we compute the velocities by integration. The vortices behind the wing are situated in the layer of air that was in touch with the surface of the wing, for they coincide with streamlines and begin at the wing. For simplicity's sake they are generally assumed to be straight and parallel to the direction of flight. They are the mathematical expression for the fact that in the mentioned layer the air particles are gliding laterally relative to each other with a finite velocity step or difference, as represented in Figure 2 at the points of the heavy line.

These vortices extend out of the airfoil, because no vortex can have an end. There are vortices in the wing parallel to the span. They must leave the wing somewhere, and thence coincide with the streamlines. Now, the number of unit vortex lines crossing each wing section, is proportional to the lift per unit length of the span. From which follows, that the number of unit vortex lines leaving per unit length of the span is proportional to the differential quotient of the lift per unit span with respect to this span.

The combination of these arguments gives the complete distribution of the trailing vortices, if the distribution of the lift along the wing span is known or assumed.

The downwash is computed from these vortices by integration. The vorticity has been shown to be derived from the velocity distribution from the vortices by integration. This integration rule is well known in connection with the electromagnetic theory and is there known as Biot Savart's rule. The velocity corresponds to the magnetic field and the vorticity to the electric current. Each vortex element contributes an ele-

ment of velocity at right angles to itself and to the radius vector connecting the vortex element with the point in question. The velocity is proportional to the length and strength of the vortex element, to the sine of the angle between the element and the radius vector and inversely to the square of the length of the radius vector. In this manner the vortices are computed from the lift distribution and the velocities from the vortices. These induced velocities give the induced drag again in the same way as we have shown in Section 37.

The vortex theory appeals to mathematicians familiar with vector analysis and accustomed to deal with singularities of functions rather than with the functions themselves. There is only one class of problems where the use of the vortex theory is more convenient than the use of the potential theory. These problems refer to the effect of the motion of a solid at a large distance from it. The influence of a deflection of the air flow on a distant point is approximated by vortices, just as the influence of displacements of air at a large distance is approximated by sources. (Section 52.) Distant effect, therefore, can be conveniently found by computing the action and interaction of vortices and sources. Such problems are not acute in aircraft design, and the reader is, therefore, referred to the literature about this method.[2]

52. Sources and Sinks

Sources and sinks belong in the same class as the vortices. They also are mathematical abstractions much used in higher hydrodynamics and most theorems for vortices hold also for sinks and sources in slightly changed form. We do not intend to go into their mathematical theory. It seems sufficient for an aerodynamic engineer to understand the logical connection between them and the plainer parts of the theory.

A source represents a continuous concentrated supply of fluid (Figure 21) at one point. Near this point, the fluid moves

[2] N. A. C. A. Technical Report No. 114· *Some New Aerodynamical Relations*, by Max M. Munk.

radially to all sides, the velocity inverse to the square of the distance from the point, so that the same volume flows through all spheres that can be drawn with the source as center. A similar continuous absorption of fluid at one point is called a sink or a negative source. Such sources do not actually occur,

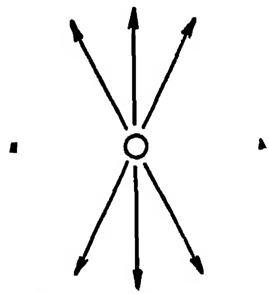


Figure 21. Point Source

but the flow in some distance from them can be descriptive of actual conditions. For instance, the flow near the bow of an airship hull. This hull pushes the air away to make place for itself, and a similar flow would happen if the vacated space would be filled by newly supplied air, rather than by the bow, produced by one or several sources located where the hull actually is, that is, outside of the airflow.

Consider one source with an intensity of supply I units of volume per unit of time, all streamlines extending radially, so that the velocity is $I/4\pi r^2$, where r denotes the distance from the source. (Figure 21.) This flow is a potential flow of the kind we have studied in the former chapters. Now superpose a flow of constant and parallel velocity V to this source. (Figure 22.) There results a flow as shown in Figure 22. Near the source the flow is not much influenced by the parallel flow, but at larger distance the parallel flow becomes more and more dominating, the radial flow subsiding rapidly. The entire flow is divided into two parts by one streamline that splits itself into

branches to all sides and divides the fluid supplied by the source from the remaining fluid. It is easy to compute the distance of the bow point A from the source, for this is the point where

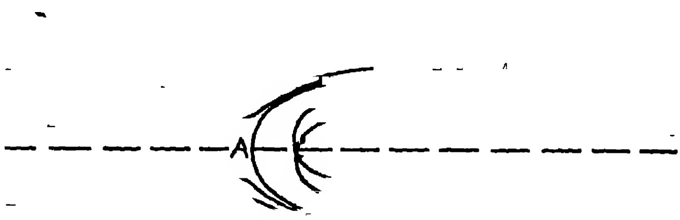


Figure 22. Superposition of Point Source and Parallel Flow

the velocity of the radial flow is just neutralized by the constant velocity V of the parallel flow. There we have, therefore,

$$V = -I/4r^2\pi$$

and hence
$$r = \sqrt{I/4\pi V}$$

It is as easy to compute the final maximum diameter of the finger-shaped surface of revolution formed by the split streamline. The supply I divided by the area of cross-section $d^2\pi/4$ must give the velocity V,

hence
$$d = \sqrt{4I/\pi V} = 4r$$

In a similar way by superposing the two flows the velocity and potential at each point can be computed. The kinetic energy of the flow outside the finger can then be computed by means of equation (10), Section 8, for the case that the finger moves through the fluid otherwise at rest with the velocity V. This finger represents then the bow of an elongated airship hull.

The velocity of flow has appreciable magnitudes near the ends only, this explains why the kinetic energy does not depend on the length of the end. The volume of apparent additional mass comes out one-eighth of the volume of the sphere with the diameter d. The conditions near the stern of the elongated hull can be treated quite similarly by means of a sink with the same strength. Hence the volume of apparent additional mass of the hull is twice the volume mentioned, one-fourth of the sphere of maximum diameter of the hull.

This, of course, holds only for very elongated hulls and only for those with a shape of the ends like Figure 22. The apparent mass so computed, however, always will give the order of magnitude with sufficiently elongated hulls, and that is often enough.

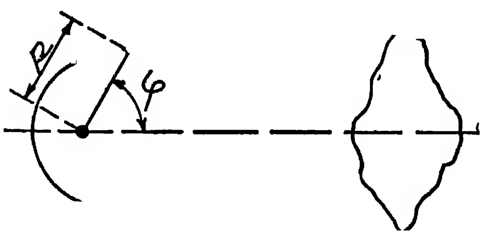


Figure 23. Diagram Referring to One-Source Bow

We insert now the short computation of the shape just discussed. We saw the intensity of supply of the point source to be

$$I = r^2\pi V$$

where r was the radius of the greatest cross-section of the hull. Let the point source be situated at the origin of a system of polar coordinates R, φ. The fluid passing in the unit of time through a spherical segment, $R =$ constant, within the cone $\varphi =$ constant is composed of two parts, one due to the constant velocity V, and the other to the point source. The first part is $(R \sin \varphi)^2\pi V$, and the second part is $\frac{1}{2}(1 - \cos \varphi)\, r^2 \pi V$. If the edge of the spherical segment coincides with the surface of the hull, the entire fluid passing, that is to say the sum of these two expressions, is equal to the intensity of the source $r^2\pi V$, whence we obtain the equation of the hull shape

$$r^2\pi V = (R \sin\varphi)^2\pi V + \tfrac{1}{2}(1 - \cos\varphi) r^2\pi V$$

or transformed

$$R = r\frac{\cos\frac{\varphi}{2}}{\sin\varphi}$$

As was to be expected, all hull shapes equivalent to one point source are geometrically similar.

Very different hull shapes can be obtained by arranging several or very many sources and sinks along the axis of the hull and by superposing their radial flows and a parallel flow.[3] The computation becomes then more and more involved, but the principle is as simple as with one source.

53. Joukowsky Sections and Karman Vortices

We close the chapters on theoretical aerodynamics with some remarks on two discoveries each connected with the name of an eminent aerodynamic scientist. Both belong in higher hydrodynamics and are beyond the scope of this book. Both, how-

Figure 24. Several Joukowsky Wing Sections

ever, are so often mentioned in literature that the aerodynamic engineer should at least know in what relation they stand to the plainer theory.

Joukowsky wing sections are curves (resembling wing sec-

[3] See reference 3, page 175.

tions) of a particular mathematical type.[4] They are interesting and useful by the fact, and by this fact alone, that it is comparatively easy for a mathematician to construct them,[5] and to find the theoretical potential flow around them for all angles of attack. Comparatively only, it is quite elaborate to determine their aerodynamic characteristics, and the proof for the correctness of the construction is very involved.

Some of the Joukowsky sections are said to be aerodynamically good. That is not essential, however. They are not actually used with airplanes, because they are too thin near the trailing edge. The upper and lower curves approach one common tangent at this trailing edge; the angle there is zero.

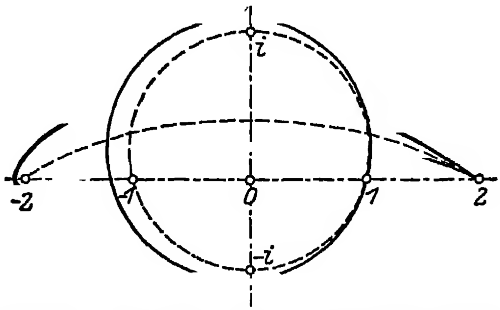


Figure 25. Joukowsky Section with the Circle Used for Its Construction

The middle curve is circular, so that the stability characteristics are not particularly favorable. The thickness and curvature can be chosen; there exists one Joukowsky section for each combination.

These sections were known before the general theory of wing sections as outlined in this book was created; that gave them a higher importance than they have now.

Karman's vortices occur in the wake behind a circular cylinder or a similarly shaped solid, moving through a liquid at right

[4] See reference 4, page 175.
[5] See reference 12, page 175.

angles to this axis.[6] If this solid is a lead pencil sticking out of the surface of water, it can be easily seen how vortices are formed on the right side and left side alternately which a short distance behind the lead pencil arrange themselves in a regular troop, marching up in two lines, not like soldiers, but each one opposite the gap of two on the other side. The ratio between the distance of two consecutive vortices and the distance of the two lines is equal for all speeds, and likewise the ratio of this distance to the diameter of the cylinder. It is about equal to five.

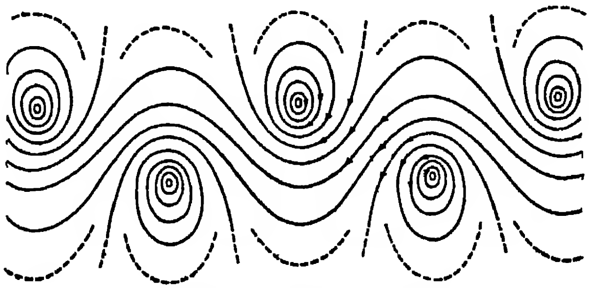


Figure 26. Karman's Vortices

Karman succeeded in computing the former ratio from the vortex theory (not treated in this book) by showing that the arrangement of vortices in two lines with equal distance between consecutive vortices is stable and can be maintained for one ratio only. This was a fine success of the theory. The importance of this computation must not be exaggerated, however. It is not possible to compute the drag of a cylinder; it has only been shown that if there are rows of vortices, they have a tendency to arrange themselves into the computed configuration.

The periodic formation of vortices behind wires and other objects explains their whistling noise when quickly moved through the air. It is not essential for the wire whether the vortices arrange themselves into a Karman configuration or not.

[6] See reference 5, page 175.

Besides, there is also a noise produced without the formation of periodic vortices. This is, of course, beyond the scope of theoretical hydrodynamics; Karman's vortices are chiefly interesting because they stand midway between theoretical hydrodynamics and the motion of solids in viscous fluids.

54. Correction of Wind Tunnel Observation

Among the several errors of wind tunnel tests, one is caused by the neighborhood of the walls, if the wind tunnel is of the closed type, otherwise by the neighborhood of the free boundaries of the jet. Its influence on the character of the air flow near the model is only secondary in most cases; the main effect is a change of the induced angle of attack and of the induced drag of wing models. This influence can be computed. It is exact enough to employ the simplest assumptions for this computation. A formula results, enabling the investigator to compute the angle of attack and the drag within an infinitely large wind tunnel from the values observed in a wind tunnel with finite diameter.

The simplest assumption is a multiplane wing with a circular front view located in the center of the circular cross-section of the wind tunnel. The radius of the multiplane wing may be r; that of the wind tunnel R. The apparent mass of a circle being twice that of its diameter, the multiplane wing is equivalent to a monoplane with the span $b = 2r\sqrt{2}$.

Introducing at once polar coordinates s and φ, where s denotes the distance from the center, the potential of the flow is

$$\Phi = (A/s + Bs) \cos\varphi$$

which expression is a solution of Laplace's differential equation for any values of the constants A and B. For the determination in a closed wind tunnel we have the condition of zero radial velocity components at the outer circle, and of these components being equal to cos φ at the points of the inner circle. This latter condition corresponds to a motion of the inner circle

with unit velocity. The radial component of the velocity is found by differentiating the potential with respect to the radius s, and comes out as

$$V_r = (-A/s^2 + B)\cos\varphi$$

Introducing the abbreviation $m^2 = 2r^2/R^2$, we obtain

$$A = -\frac{2r^2}{2 - m^2} \qquad\qquad B = \frac{-m^2}{2 - m^2}$$

The area of additional apparent mass at last is computed by means of the integral

$$-\int_0^{2\pi}\left(\frac{A}{r} + Br\right)\left(-\frac{A}{r^2} + B\right)\cos^2\varphi r d\varphi = -r^2\pi\left(B^2 - \frac{A^2}{r^4}\right)$$

Substituting the values for the constants A and B, the area of apparent mass of the flow between the two circles results

$$\pi r^2 \frac{2 + m^2}{2 - m^2}$$

The total area of apparent mass including the area of the inner circle is, therefore,

$$\pi r^2 \frac{4}{2 - m^2}$$

This is equal to the apparent mass of a circle moving in unlimited air, multiplied by the factor

$$\frac{2}{2 - m^2}$$

Hence the observed induced angles of attack and the observed induced drag must be multiplied by the last expression in order to obtain the values corrected for the wall influence.

In a free jet wind tunnel, the condition at the jet boundary

is a zero potential, for the impulsive pressure creating the two-dimensional flow is zero at the outer boundary of the jet. The condition at the inner circle remains the same. The constants result now

$$A = \frac{-2r^2}{2 + m^2} \qquad B = +\frac{m^2}{2 + m^2}$$

giving an apparent mass

$$\pi r^2 \frac{4}{2 + m^2}$$

Hence the apparent mass is now decreased.

Since the computation is only exact to the first order anyway, it is permissible to simplify the correction factor by performing the division and by using the first term only.

$$\frac{2}{2 - m^2} = 1 + \frac{m^2}{2} \pm \dots$$

$$\frac{2}{2 + m^2} = 1 - \frac{m^2}{2} \pm \dots$$

$1 + \frac{1}{2}(b/D)^2$ and $1 - \frac{1}{2}(b/D)^2$ are, therefore, the respective correction factors for monoplane models. The correction factor must be taken equal for models with equal area of apparent mass of the front projection.

55. Distribution of Thrust Along the Propeller Blade

The energy losses of the propeller depend noticeably on the distribution of the thrust over the length of the blades, and these losses can be diminished by a favorable distribution. The two secondary induced losses, the loss due to the finite number of blades and the loss due to the rotation of the slipstream, call for a gradual decrease of the thrust per unit of propeller disc area towards the inner and outer ends of the blades. Near the center the thrust is naturally less dense, and hence, the loss from rotation is generally kept reasonably small without special effort of the designer. No further improvement is here

possible. The width of the blades, however, is not always as much tapered towards the tips as would be desirable for keeping the loss due to the finite number of blades small. In spite of it the actual thrust distribution is almost as favorable and the density of thrust always decreases properly, because the wing automatically produces a small density of lift quite close to its end. Still, the wing works then under less favorable conditions and with smaller efficiency, and the weight of the propeller and the centrifugal force also are unnecessarily great. It is, however, sufficient to keep in mind that the wing tips must be round, and then to consider the two chief energy losses only, the energy absorbed by the air friction and the slipstream loss proper.

A small variation of the distribution of the thrust hardly changes the entire loss noticeably, especially if the distribution is already close to the best distribution. Hence, the problem is less the exact determination of the best distribution of thrust than the derivation of a simple expression which gives quickly an idea as to how the thrust must be distributed.

The conditions are quite different from those for ordinary wings. There, the inductive losses form a much greater part of the entire losses, and the remaining part, the friction loss, is entirely independent of the lift distribution. Hence, with ordinary wings the distribution of the lift is determined by the consideration of the induced drag exclusively. With the propeller, however, the friction is the dominant part, and both, not only the slipstream loss but the loss of friction as well, depend on the distribution of the thrust, for the velocity of the blade elements relative to the air is variable and is greater farther from the axis. And whether the ratio lift/drag of the blade elements is constant or not, the ratio of the useful work done by the lift to the energy absorbed by the drag is quite different from it and certainly not constant in general. The useful work is done in the direction of the constant velocity of flight, but the friction is multiplied by the relative velocity of

the blade element and the smaller the relative velocity, the smaller is the loss. Therefore, the consideration of the friction alone calls for a great density of the thrust near the center. The slipstream loss alone, however, calls for constant density of the thrust for area of propeller disc. Hence, the smallest loss occurs at a compromise distribution, that is, that the thrust is neither concentrated near the center nor distributed evenly, but its density decreases uniformly towards the outside.

A short calculation will be sufficient to give numerical information on the desirable decrease. Let C_T denote the density of the thrust per unit of the disc area divided by the dynamic pressure of the velocity of flight, that is,

$$C_T = \frac{dT}{V^2\rho/2 \cdot df}$$

where dT denotes the infinitesimal thrust acting on the small area df of the propeller disc. C_T may be assumed to be small, say up to .50. Then the slipstream loss originated by this thrust is approximately $\frac{1}{4}$ C_T multiplied by the useful work done by the same element of thrust. The density of drag measured in the same way is $C_T \cdot C_D/C_L$ and the work absorbed by this drag is $C_D/C_L \cdot v/V$ times the useful work, where v denotes the velocity of the blade element relative to the air and V the velocity of flight. For simplicity's sake, we replace the velocity v by the tangential velocity of the blade element, which is somewhat smaller but not so much that this would greatly injure the final result.

The problem can now be stated in the following way. Let r denote the radius of the blade element, that is, its distance from the axis. The entire thrust is easily found to be

$$T = 2\pi q \int_0^{D/2} C_T r\, dr \tag{1}$$

The slipstream loss for the path of length l of the airplane can be taken as

$$2\pi q \frac{l}{4}\int_0^{D/2} C_T r dr \tag{2}$$

The energy absorbed by the friction during the same time is

$$\frac{(2\pi)^2 qn}{V}\int_0^{D/2} \frac{C_D}{C_L} C_T r^2 dr \tag{3}$$

The coefficient of thrust density C_T variable along the length of the blade, is to be determined in such a way that for a given thrust (1) the sum of the two losses (2) and (3) becomes a minimum.

The solution is easily obtained, using the principles of calculus of variations. It can be seen that by transferring an element of thrust from one place to another the entire loss must not be changed, nor the entire thrust, and hence, the ratio of the local change of loss to the change of thrust must be equal to a constant, say λ. The condition which determines the desired function is, therefore,

$$\text{Variation of } [\lambda(1) + (2) + (3)] = 0$$

where the variation originates by any variation of C_T. That is the same condition as if

$$2\pi q \int_0^{D/2} \left(\tfrac{1}{4} C_T^2 r + \frac{2\pi n}{V}\frac{C_D}{C_L} C_T r^2 + \lambda C_T r\right) dr$$

is to be made a minimum. The variation is proportional to the integral over

$$\tfrac{1}{2} C_T r + \frac{2\pi n}{V}\frac{C_D}{C_L} r^2 + \lambda r \doteq 0$$

which then has to be zero at every point r. Hence the solution of the problem is

$$C_T = 2\lambda - 4\frac{\pi n}{V}\frac{C_D}{C_L} r \tag{4}$$

It shows, as was to be expected, that the thrust density must be constant only if the drag coefficient of friction $C_D = 0$. But if the friction is taken into account, it is not constant but has to decrease towards the outside. The simplifying assumptions make it appear as a linear function of the radius, and that is just what we wanted, and exact enough. It remains only to determine the value of the constant from the condition that the entire thrust has a particular value in order to obtain the final expression for the density of thrust. By substituting equation (4) in equation (1) it appears that

$$T = 2\pi q \int_0^{D/2} \left(2\lambda - \frac{4\pi n}{V}\frac{C_D}{C_L}r\right) r\,dr$$

i.e.,

$$T = 2\pi q\lambda \frac{D^2}{4} - \pi q \frac{\pi n}{3V}\frac{C_D}{C_L}D^3$$

Hence

$$2\lambda = \frac{T}{D^2\pi/4 \cdot q}\,\frac{4}{3}\frac{C_D}{C_L}\frac{\pi n D}{V}$$

and finally,

$$C_T = \frac{T}{D^2\pi/4 \cdot q} + \frac{4}{3}\frac{C_D}{C_L}\frac{\pi n D}{V} - 2\frac{C_D}{C_L}\frac{\pi n \cdot 2r}{V}$$

At a radius 2/3 of the greatest radius, the thrust has the mean density, the same that it had to have without friction. At the blade tips, the density must be

$$\frac{T}{D^2\pi/4 \cdot q} - \frac{2}{3}\frac{C_D}{C_L}\frac{\pi n D}{V} \tag{5}$$

This expression can become negative. But then one essential assumption for the proceeding is no longer valid: C_L/C_D is no longer constant but even changes its sign. For this reason a negative thrust appears uneconomical in accordance with what everybody expects. Equation (5) becoming negative rather indicates that the diameter has passed beyond its most economical value. This value appears, then, to be

$$D^3 = TV/nq \cdot C_L/C_D \cdot 6/\pi^2$$

Let us see now by an example, what the derived formulas give for usual conditions. They will give different results for the same propeller under different conditions of flight, and this fact is an additional reason for confining the calculation to a moderate degree of exactness.

Illustration:

Diameter = 9 ft., that is, disc area 63.5 sq. ft.

n = 25 revolutions per second, giving $\pi nD = 706$ ft./sec.

Thrust = 400 lb.

V = 100 mi./hr.

$C_L/C_D = 22$

q = 25 lb./sq. ft. dynamic pressure of flight

$$D^3 = \frac{400 \times 100 \times 1.47 \times 22 \times 6}{25 \times 25 \times \pi^2} = 1{,}260 \text{ ft}^3$$

$$D = 10.8 \text{ ft.}$$

This value is close to the actual diameter. However, the conditions of great velocity are favorable for a small economical diameter. Suppose on the other hand the velocity of the same propeller to be 60 mi./hr. only and the thrust to be 575 lb. The dynamic pressure q may be 10 lb./sq. ft.

Now,

$$D = \frac{575 \times 60 \times 1.47 \times 22 \times 6}{10 \times 25 \times \pi^2} = 2{,}700 \text{ ft}^3$$

$$D = 14 \text{ ft.}$$

Even in this case, the diameter results only 14 ft., which indicates that the improvement theoretically possible cannot be exceedingly great.

In the first case the mean value of C_T appears .252 and C_T at the blade tip has to be .106. The mean value agrees with the desired value at ⅔ of the radius as always. That is a rather great variability, which corresponds to the fact that the diameter is only slightly smaller than the most economical diameter. In the second example, the mean value of C_T is .90. This is comparatively high and in consequence of it the developed formulas give too large a diameter, for the factor .25 for the

induced loss is too high for the large values of C_T. Paying no attention to this, the coefficient at the blade tips appears .658. The density is much more constant now, according to the greater predominance of the slipstream loss.

The results obtained are only approximate. The formula for the diameter is not to be considered as a literal prescription. The weight of the propeller is not considered nor the resulting tip velocity, which cannot be increased without limit. Besides, the diameter is more often determined by the general lay-out of the airplane. The formula is only to show whether an increase of the diameter means an improvement at all, and to give an indication of how much.

56. Synopsis

The vortices and sources are special mathematical conceptions used to represent the same theoretical flows, ordinarily represented by the velocity potential.

Joukowsky wing sections have a theoretical flow more easily computed than any other sections. Karman vortices form wakes behind certain obstacles.

The induced drag measured in wind tunnels must be corrected by a factor $1 \pm b^2/D^2$, where plus refers to closed wind tunnels, minus to open jet tunnels, b denotes the span, and D the tunnel diameter.

The thrust distribution along propeller blades for smallest power required cannot be computed from the slipstream loss alone, but also depends on the friction losses.

CHAPTER VIII

AIR FRICTION

57. Outlook

A broad discussion of the motion of the air under the influence of friction is of little real use. We discuss as briefly as possible a few terms often found and enter into the practical part of the theory,—the model rules. The simple model rules are only approximations, but as such they are of immense practical use, and they refer to all kinds of flows, not only to theoretical ones. They are used by employing coefficients of the air forces, assumed to be constant.

The model rules that include friction deal with the variations of these coefficients. They occur chiefly in connection with the interpretation of model tests, a subject of great practical importance.

The discussion of the surface friction belongs properly in this general chapter on friction. Our knowledge of the surface air friction is wholly based on experience, but the model rules suggest a convenient formula for its magnitude.

58. Discussion of Terms Often Found

The flow produced by solids moved through air differs from the theoretical potential flow in a perfect fluid, particularly if the solid is blunt. Blunt shapes leave a strong wake of moving fluid behind them. They experience a correspondingly large drag, used for providing the kinetic energy of the wake. Such shapes, therefore, are unsuitable for aeronautic design.

The wake can be made visible by means of smoke. Portions of the wake are then seen rotating in whirls that are sometimes called vortices. They must not be confounded with the vortices

of the vortex theory. The latter denote an angular velocity of a fluid element, a whirl is a rotation of a finite portion of the flow.

The flow in the wake of a blunt body is very different from the type of the potential flow down to the smallest regions. A potential flow is laminar, which means each layer (lamina) of the fluid continues to form a layer. The flow in the wake, however, is turbulent, as can be observed in the flow of smoke

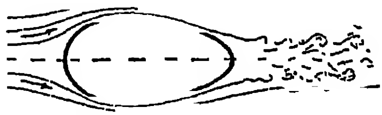


Figure 27. Wake Behind a Blunt Body

leaving a smokestack. Each portion of the smoke continuously breaks up into whirls, these separate again into smaller whirls; a group of small whirls joins together forming a larger whirl that immediately breaks up into smaller ones again, and so on, consequently the whole air is both in continuous motion and relative motion. There are no definite velocities of flow at a point, not even momentarily. We are still entirely ignorant about the dynamic principle from which the turbulence arises, nor have we any detailed knowledge about the type of motion.

Streamlined bodies, pointed or finger-shaped in the rear, have much smaller wakes. The larger part of the flow is a potential flow, and not very different from the theoretical potential flow belonging to the shape. The air forces also are not very different from the theoretical air forces; they are only slightly modified by the wake and a comparatively small drag is produced that provides the energy for the small wake. This wake extends really from the leading front of the body. It is formed along the surface, where the air flowing along the solid walls is particularly affected by the friction forces. The air particles next to the wall are torn away by the solid passing by, and im-

pelled forward with the motion of the wall. They in turn impel the next particles with them, and it can easily be seen that in this way a layer of air accompanies the solid, becoming thicker towards the trailing end of the body. This is the so-called boundary layer of Prandtl.[1] The flow in the boundary layer is laminar in some cases and turbulent in others; it may also be turbulent over a portion only. The boundary layers on all sides of the body join together at the rear thus forming the wake.

The boundary layer theory deals with the flow inside of the boundary. Special kinds of aerodynamic theory have been developed for both laminar and turbulent boundary layers. They have led to a clearer insight of what is going on, but it has not been possible to compute the drag of any solid. The theory, however, has given some results regarding the surface friction of plates. They agree with the observed friction.

The computations relating to boundary layers are decidedly useless to the designer. However, it is useful for him to know about boundary layers. They play an important part in the formation of the whirls. The theoretical and actual pressure at the trailing end is larger than some distance ahead. This larger pressure may force the air in the boundary layer into a forward motion larger than the velocity of motion of the body. The boundary layer is then swelling out, the swollen portion distorts the entire flow and becomes a whirl, separating itself from the surface. This cannot happen on the front, as there the pressure is decreasing in the direction of the flow and tends to diminish the thickness of the boundary layer. This is why a blunt front portion is more permissible than a blunt rear portion, and why the good streamline shapes are drop-shaped. It also explains why the breakdown of the smooth flow occurs on the top side of a wing rather than on the bottom. It is on the top that we have the large pressure gradient from front to back.

It has been demonstrated that the flow can be materially modified by influencing the air near the surfaces, that is, the

[1] See reference 10, page 175.

boundary layer. A rotating cylinder experiences a large lift. The flow is unsymmetrical because the friction conditions are entirely different on the two sides. This was long known, and has been practically employed by Flettner, although not for aircraft. Other investigators have tried either to remove the boundary layer by sucking or pumping it off, or to get rid of it by pressing air into it through fine openings in the surface. These schemes have no practical importance at present.

With large airships, the thickness of the boundary layers becomes considerable. Instruments for the measurement of the velocity of flight must be mounted far enough from the surface of the hull to be outside of the boundary layer. There is then a case where the reality of the boundary layers becomes striking and where it is useful for the engineer to know the thickness of the boundary layer.

59. The Square Law

The experiences with regard to air forces, gained from tests or from practice have to be combined to new conclusions for each new design. The rules for the computation of air forces and pressures from older experience are called model rules. In many cases the designer goes back to wind tunnel tests, and the objects exposed to the airflow of wind tunnels are generally called models. Model rules were chiefly used for the interpretation of model tests.

A specific model rule will be discussed in the next section. The most general model rule in aerodynamics is the so-called square law. Suppose two solids differing in scale only to move at the same angles of attack through two media of different density at different velocities. The square law expresses that all corresponding air pressures are proportional to the dynamic pressure of the motion, and hence all corresponding air forces are proportional to the density, to the square of the velocities, and to the square of the scale, that is, to corresponding areas.

The square law is also applied to objects performing rotating

motions, for instance, to propellers. The angles of attack at corresponding points are equal only if the ratio of the tip velocity of the propeller to the velocity of flight is the same. The forces are then proportional to the squares of the velocity, not merely to the square of the r.p.m. Hence, if a propeller is enlarged to twice the scale, and the speed of revolution is kept the same, the thrust is nevertheless sixteen times enlarged, since both the areas and the dynamic pressure of the tip velocity are multiplied by four each.

The square law is the very base of all aerodynamic computations. Most designers prefer to apply it in an indirect way, by using coefficients. That saves computation work and is moreover more instructive. The air forces and pressures are written as products of several factors, one depending on the shape of the solid only, and the others on the air velocity, density, and the size of the solid. The shape factor is equal to the quotient:

$$\frac{\text{Air Force}}{\text{Area} \times \text{Dynamic Pressure}}$$

It is at once computed after a test has been finished, and the result is recorded or published, by employing this ratio. It is called the air force coefficient. The coefficient is truly defined only after an agreement has been reached about the choice of the area in the denominator. Even then, there are several kinds of coefficients in use, differing by factors. In some countries, and formerly in the United States as well, a coefficient half as large as given above was in use. There are also coefficients still in use that are odd multiples of the above coefficient. The latter is called the absolute coefficient because the quotient gives the same coefficient regardless of the kinds of units used to express the quantities, provided the units are consistent. The same coefficient results whether sq. m., kg. and kg./sq.m. are used or sq. ft., lb., and lb./sq. ft. The epithet "absolute" refers to the definition of the coefficient rather than to the coefficient itself.

The validity of the square law is equivalent to a constant

value of the coefficient whatever the scale, the velocity, or the fluid may be. As a matter of fact, tests under different conditions do not always give the same coefficient. The square law is not strictly true. Even then, the coefficients are much less variable than the air forces themselves, and it is, therefore, much easier to work with them.

Theoretical potential flows comply rigidly with the square law, but the compliance with the square law is by no means restricted to potential flows. Flat discs, for instance, moving face ahead and producing a strong wake have a drag following the square law remarkably well.

The square law is really based on the similarity of the two flows, whether much different from the theoretical potential flow or not. Two flows are called similar if all streamlines are similar and all velocities proportional. The mass forces of all corresponding elements obey then the square laws, as follows directly from the fundamental equations of dynamics, and hence, the resultant forces, being their effects summed up, do likewise. The summed up effects of the friction forces are generally small when compared with those of the mass forces.

Constant coefficients, therefore, are an indication of similar flows. Conversely, we are only entitled to expect the square law to hold if we have a right to expect the flows to be similar.

60. Reynolds' Model Law

Since the square law is only an approximation and the coefficients are at least slightly changeable, the problem arises as to what laws there are governing the change of the coefficients. More particularly, can the coefficients be expected to change with any simultaneous change of the scale, the velocity and the fluid, or can one change be made up by another, so that the new combination will necessarily give the same coefficient? This is an important question, for if so, it would be possible to make model tests which are certain to yield reliable results, and the

determination of all possible coefficients of one shape will be much simplified.

A few remarks on the air friction are necessary before we can proceed with this question, because the air friction is the only force besides the mass forces. The mass forces alone would give rise to the square law, the effect of combined mass forces and friction has to be examined. The air friction is the resistance of the air to the rate or velocity of angular distortion, not to this distortion itself, like the shear forces in solids. All experiments have shown that otherwise the air friction follows exactly the ordinary relation known for shear. The friction force is directly proportional to the rate of gliding and to the area; the factor of proportionality depends on the fluid and on its physical conditions. Thus, if all velocities of a fluid are parallel and further proportional to the distance from one parallel plane, the friction forces acting along plane surfaces parallel to this plane are proportional to the size of the surface and to the gradient of the velocity. They can, therefore, be written:

$$\text{Friction Force} = \text{Area } dV/dx \cdot \mu$$

where μ, called the modulus of friction, depends on the kind of fluid and on its physical conditions. The rate of gliding, dV/dx, is a kind of angular velocity. The physical dimension of the friction modulus is seen to be (Force $\times$ Time/Area). For the application, this modulus divided by the density is of importance and has received a special name. The kinematic modulus of viscosity, $\nu = \mu/\rho$ has the simple physical dimension Area/Time.

We arrive at the answer to our question regarding equal coefficients under different conditions, if we compare two similar flows. The one is a physical flow, the other is only hypothetical, and we inquire whether it is physically possible.

We regard in turn the mass forces, the friction forces and the pressure in the hypothetical flow separately and see whether they will be in equilibrium. The mass forces of each element of fluid compare with the mass forces of the corresponding ele-

ment in the similar and physical flow in keeping with the square law. If the friction forces would also be in keeping with this law, the pressure which is the resultant of all friction forces and mass forces would necessarily follow. Whence we conclude that the hypothetical flow is physically possible, and furthermore, is complying with the square law, if all friction forces follow the square law.

Now the friction forces are increased proportional to LV, where L denotes a characteristic length and V a characteristic velocity, whereas the mass forces are increased proportional to $L^2V^2\rho$. Hence, both are increased in the same ratio, if

$$LV\mu = L^2V^2\rho$$

or if

$$R = \frac{LV}{\mu/\rho} = \frac{LV}{\nu} \text{ remains constant.}$$

The last expression is called Reynolds Number, but should be called Reynolds parameter.[2] Different combinations of scale, velocity, and fluid give the same coefficient if Reynolds parameter is the same. Its form indicates that a smaller scale can be compensated by a larger velocity or by a smaller kinematic viscosity of the fluid, or by certain combinations of changes of these two. If Reynolds Number is different, the coefficients cannot be expected to be equal, but they must be determined separately. On the other hand, it is unnecessary to determine the coefficients for different combinations having the same Reynolds Number. The result of a complete investigation of a coefficient of a special geometric shape is naturally laid down by plotting or tabulating the coefficient against the Reynolds Number. A larger Reynolds Number means smaller viscosity, all other things being equal, but it does not necessarily mean smaller friction forces. The Reynolds Number really has no physical meaning beyond being a parameter. Its magnitude depends on the arbitrary choice of the characteristic length and

[2] See reference 11, page 175.

velocity, and every mathematical function of it could be used in the same way.

The problem how the coefficients change with the different parameters has so been reduced to the problem of how they change with the one Reynolds parameter. No general theory about this latter problem exists; the answer is almost entirely left to experimental investigation.

61. Surface Friction

The most important effect of the viscosity of the air is the creation of a drag on any solid moving through it. We saw that the creation of lift is always accompanied by a drag called the induced drag even in perfect fluids. In actual air, all bodies experience a drag while wings experience a drag larger than the induced drag. This drag, or with wings the portion of it in excess of the induced drag, is properly called friction drag, as it is intimately connected with the internal friction of the air.

The friction drag must not be understood, however, to be transferred to the moving body by friction merely, being the resultant of all tangential air force components on the surface of the solid. The friction drag is larger than that—it is the resultant of both the tangential and normal air force components. The pressure distribution in viscous air itself has a drag component, which adds to the drag component of the tangential forces. Both together form the friction drag.

This division of the friction drag into two parts, separating the effect of the tangential forces from the effect of the normal pressures is well defined in itself, although it is often difficult to make the separation. Nor is the designer much interested about the division as far as the effect is concerned—both kinds of drag are equally undesired. Nevertheless it is useful to divide the drag that way into a form resistance and a surface resistance, because the latter, the effect of the tangential air forces, can often be estimated. That gives then the minimum of the drag to be expected. If the surface drag forms a considerable portion

of the friction drag, all efforts to diminish the friction drag by more elaborate streamlining appear futile at the beginning, and the knowledge of the magnitude of the surface drag may restrain the designer from over-streamlining and increasing the drag thereby.

Pure surface drag is measured on plane surfaces. It depends on the kind of boundary layer, whether laminar or turbulent, and is not always alike under almost equal conditions for that reason. The boundary layers along surfaces of aircraft parts are generally in turbulent motion. The coefficient of the surface friction can then be expressed as a function of the Reynolds Number. The relation is derived from observations; theory provides some interesting comment on the form of the expression, but is unable to figure it out without recourse to experience.

The coefficient of surface friction C_F is defined by

$$C_F = F/Sq$$

where F denotes the friction force, S surface of the plate (both sides) and q the dynamic pressure corresponding to the velocity of motion. The Reynolds Number is formed by choosing the length of the plate in direction of the motion as the characteristic length. The surface friction coefficient is expressed by:

$$C_F = 0.074/\sqrt[5]{R} \qquad R = Vl/\nu$$

The application of this formula to the computation of the surface resistance coefficient of curved surfaces is not strictly justified, but it leads to useful estimates. Let us apply the expression to the case of an airplane wing with a chord of 200 cm., moving through the air with a velocity of 4,000 cm./sec. The modulus of viscosity of air depends on its temperature, and is about 1/7 sq. cm./sec. under average conditions. This value must be combined with the velocity and length in cm./sec. and cm. respectively. The Reynolds Number for the wing is, therefore, $R = 200 \times 4{,}000 \div 1/7 = 5{,}600{,}000$. Inserting this Reynolds Number into the expression for the surface drag coefficient

gives the value $C_F = 0.074/\sqrt[5]{5,600,000} = 0.0033$. The drag coefficient of wings is generally referred to their projected area, and this being about half as large as their surface, the drag coefficient of surface friction would be twice that large.

62. Influence of the Compressibility

The pressures at the propeller tips are much larger than at all other points of the aircraft. The question arises about the influence of the compressibility of air on the propeller forces. Such influences are indeed appreciable, but the theoretical study of this question has not yet yielded useful results. Any detailed discussion about the effects of compressibility therefore seems out of place.

Even in a perfect fluid, a drag is possible if the velocity of flow reaches the velocity of sound of the fluid. This happens before the velocity of motion becomes that large. Propellers have actually been used with a tip velocity reaching the velocity of sound without a drop in the efficiency. It seems then that this drag caused by compressibility is not necessarily large.

The velocity of sound is the velocity of propagation of any pressure differences, and this is why it is the critical velocity for compressibility effects. Incompressible fluids have an infinite velocity of sound, and fluids can be considered as incompressible so long as the ratio of their velocity of sound to the largest velocity of flow is very large. This ratio is characteristic for the effect of the compressibility and plays a part similar to that of the Reynolds Number in the theory of viscosity effects.

63. Synopsis

The square law is only approximately correct. It expresses that all corresponding air forces vary with the square of the velocity and with the square of the linear scale, but directly with the density, if these three change and everything else remains as it was.

Reynolds model law expresses the condition for the strict

correctness of the square law. The square law is strictly true, if in the two cases compared the Reynolds Number has the same value. This number is computed for both cases; it is equal to:

$$\text{Reynolds Number} = \frac{\text{Length} \times \text{Velocity}}{\text{Kinematic Viscosity}}$$

The coefficient of the surface friction is equal to this friction per unit surface, divided by the dynamic pressure of the motion. This coefficient for turbulent flow has the magnitude

$$C_F = 0.074/\sqrt[5]{R}, \qquad \text{where } R = Vl/\nu$$

64. Problems and Suggestions

1. The drag of an airplane strut is 1 lb. per running foot. The cross-section is enlarged to twice the original scale. How large is the drag per running foot, everything else remaining unchanged?

2. An airplane flies at sea level at a velocity of 100 miles per hour. What is its velocity at an altitude of half the density of air, if it flies there at the same angle of attack?

3. How does the horsepower absorbed at that height compare with the horsepower absorbed at sea level?

4. By what fraction has its weight to be diminished if the horsepower at half the air density has to be the same as at sea level?

5. Two similar propellers on a test stand have the diameters 8 ft. and 11 ft. What is the ratio of their horsepower absorbed, if both run at the same r.p.m.?

6. Compute the Reynolds Number for a wing with a 10 ft. chord, flying at a speed of 90 mi./hr., taking the chord as characteristic length, if the kinematic viscosity of the air is 1/50 in^2/sec.

7. The viscosity of air is independent of its density, hence, its kinematic viscosity is inverse to its density.

8. A model at a scale 1 to 10 is tested in compressed air at

the true flying velocity. How large must be the density to obtain the true Reynolds Number?

9. How large has the density to be if the speed will be so adjusted as to obtain 4 times the true pressures?

10. A model tested in atmospheric air at the true Reynolds Number experiences the true air forces, but not necessarily the true air pressures.

11. Compute the surface friction of a vane 2 ft. in length and 2 ft. in height, moving through air with the density 1/420 lb. sec^2/ft^4 and the kinematic viscosity 1/50 $in^2/sec.$ at a velocity of 80 ft./sec.

12. Compute this friction in air of twice this pressure.

CHAPTER IX

MEASURED AIR FORCES

65. Outlook

In this last chapter we introduce the reader into the study of the actual air forces. We discuss the more important diagrams used and coefficients employed. The main results of the theory appear in a particularly simple form when written as relations between the coefficients. The theoretical air forces are compared with the true air forces and some broad statements added about the magnitudes of the latter, and the test methods.

The aim of this last chapter is the same as the aim of the whole book: to assist the reader in obtaining a firm hold of the immense amount of empirical information on air forces collected during the last fifteen years.

66. Airship Forces

We are dependent on tests with actual airships for the determination of the airship forces. Wind tunnel tests are too unreliable on account of the small model scale required. The airship drag is either computed from the absorbed horsepower or it is determined from deceleration tests. At full speed of the airship all engines are suddenly stopped, and the rate of decrease of its velocity is observed. The decelerating force is computed therefrom by means of Newton's law and the parasite drag of the stopped propellers subtracted. The gross drag so obtained contains the drag of the accessory parts as the cars, radiators and fins.

The mass to be substituted into Newton's equation is substantially the mass of the airship, which again is equal to the mass of the displaced air, since the ship has to be in static equilib-

rium, with all its weight carried by buoyancy. The additional, apparent mass of the hull must be added,—only a few per cent of the entire mass.

The observed velocities are evaluated by plotting their reciprocals against the time. This follows from the square law. The drag being proportional to the square of the velocity, it can be written in the form $D = AV^2\rho/2$, where A, the area of drag, is approximately constant. Hence, K denoting the mass of the ship, the motion is determined by the equation

$$-K\, dV/dt = AV^2\rho/2$$

The solution of this differential equation can be written:

$$V = \frac{2v/A}{t}$$

where v denotes the effective volume

From which $1/V$, the reciprocal of the velocity, follows to be proportional to the time, measured from a suitable origin. Hence, when plotted against time, it should give a straight line. The slope of this line is equal to the area K of the drag, divided by twice the effective volume of the airship, the latter including the volume of the additional apparent mass.

Since the reciprocal of the velocity is used, it is practical to determine the velocity by means of anemometers rather than by taking air pressure measurement. Such anemometers, a kind of small windmill, can easily be constructed to record the time elapsed between equal numbers of revolutions of the anemometer wheel. This time is proportional to the reciprocal of the velocity. Anemometers are better suited for recording small velocities than pressure instruments, as their effect is directly proportional to the velocity, not to its square.

Figure 28 shows a plot of the reciprocal velocity against the time as obtained for the Zeppelin Airship L 70. It appears that the observed points arrange themselves along a broken line consisting of two straight lines. This indicates that the drag co-

efficient suddenly changed during the test, increasing to a larger value. The value observed at the larger velocities is the one of practical interest.

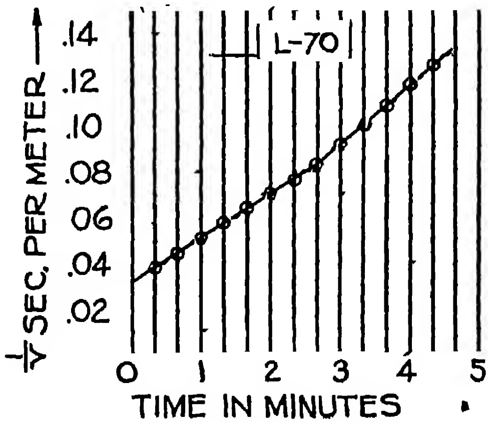


Figure 28. Plot of Airship Deceleration

The drag of airship hulls is expressed by a drag coefficient

$$\frac{\text{Drag}}{\text{Area} \times \text{Dynamic Pressure}}$$

where the area is often chosen as the ⅔ power of the volume. That seems to be a little far-fetched, but the coefficients show the relative drags of different hull forms of equal buoyancy. The magnitude of these coefficients for Zeppelin ships was about 0.02. Table III gives the values for several types. In this table, d and l denote the maximum diameter and the length. It is interesting to compare the fourth column, the area of drag, with the one computed from the expression for the friction drag. For this estimate the surface is assumed to be ⅞ of the circular cylinder with the diameter d and length l, which fraction includes the fin areas and C_D is taken 0.0014. The computed values are given in the fifth column of Table III. The surface friction is seen to be about half the total drag. The last column

of the table gives the coefficient computed by dividing the total drag by $(\text{Vol.})^{2/3} \times V^2\rho/2$.

TABLE III. DRAG OF RIGID AIRSHIPS

Name of Ship	Diameter d, ft.	Length l, ft.	Total Drag Area A, sq. ft.	Friction Drag Area sq. ft.	Coeff. Total Drag C_D
LZ10	45 9	460	565	81	0 069
L 33	78 3	645	409	194	025
L 36	78.3	645	519	194	.029
L 43	78.3	645	525	194	030
L 44	78.3	645	371	194	.020
L 46	78.3	645	364	194	020
L 57	78 3	745	445	224	.022
L 59	78.3	745	425	224	.021
L 70	78.3	694	405	209	.020

We proceed now to the moments. A paradox though it may seem, the airship hull theory is supported by tests with models of hulls having fins, rather than with bare models. This, however, gives a practical value to the theory, as airship hulls without fins are not used. The theory gives a moment of the air forces only, no resultant lift. Bare hull models show a lift when inclined towards the direction of the air stream in the wind tunnel. This indicates the pressure distribution. Hull models with fins likewise give a lift. This lift now can be attributed to the fins

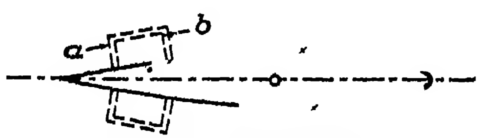


Figure 29. Airship Model

and to the portions of the hull influenced by the fins, and then the resultant moment of the pressures over the remaining portions of the hull agrees with the one predicted by theory.

This is demonstrated by the following model test, at which the lift and the moment with respect to the center of volume were measured. The model, represented in Figure 29, had a length of 1,145 mm., a maximum diameter of 188 mm. and a volume of 0.0182 m^3. Two sets of fins were attached to the hull, one after the other. The smaller fins were rectangular,

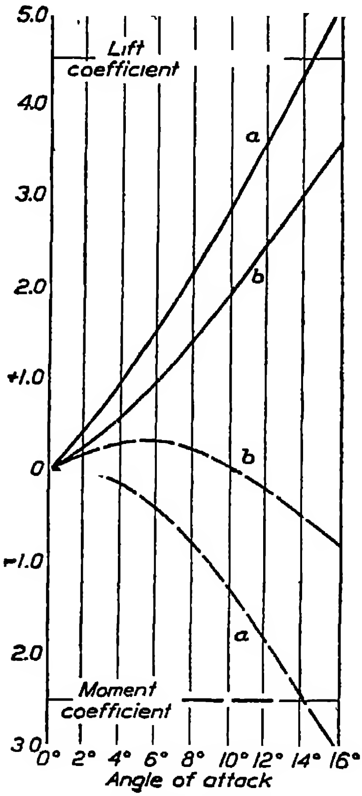


Figure 30. Plot of Airship Forces

6.5 by 13 cm., and the larger ones, 8 by 15 cm. The two-thirds power of the volume is 0.069 m^2. In Figure 29 both fins are shown. The diagram in Figure 30 gives both the observed lift

and the moment, expressed by means of coefficients. The lift is divided by the dynamic pressure and by the two-thirds power of the volume; the moment is divided by the dynamic pressure and by the volume.

If the difference between the observed moment and the moment computed by means of the theory is computed, and this difference divided by the observed lift, the apparent center of pressure of the lift of the fins is obtained. Diagram Figure 31 shows the position of the center of pressure so computed. The two horizontal lines represent the leading and the trailing end of the fins. It appears that for both sizes of the fins the curves nearly agree, particularly at the larger angles of attack, at which the tests are more accurate. The center of pressure is situated at about 40% of the chord of the fins. Without this lift, concentrated in the vicinity of the fins, the air forces would then constitute a pure couple of the magnitude given by the theory.

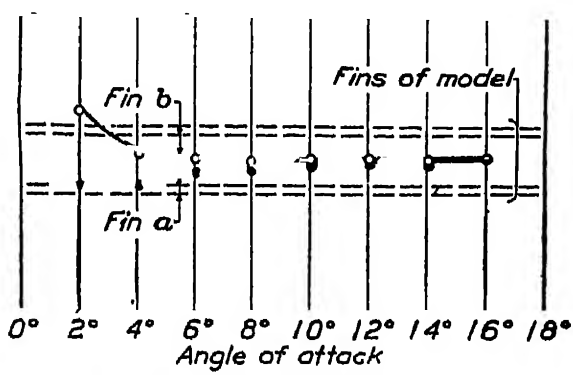


Figure 31. Center of Pressure of Fin Forces

67. Wing Forces

The determination of the aerodynamic characteristics of wing sections has formed the main portion of the wind tunnel research of the past years, and it seems that it still does so. The steady flow of incoming information about wing forces has assumed

definite shape; standard practices have developed with respect to the coefficients used and the choice of diagrams adopted for their representation. An intimate knowledge of them is indispensable to the aerodynamic engineer, together with the main theoretical relations between these coefficients.

The tests are generally made with rectangular wing models having equal and parallel sections along the span, and an aspect ratio of this span to the chord of 5 or 6. The lift coefficient, drag coefficient, and moment coefficient are obtained by dividing the force by the wing area and by the dynamic pressure; the moment is divided besides by the length of the chord. This moment refers ordinarily to the leading edge, but lately some papers have come out where it referred to a point at the chord 25% behind the leading edge. In agreement with theory this point gives a moment coefficient almost constant over the used range of the angle of attack.

In some foreign countries, and formerly in the United States as well, half of the above coefficients are in use; they were originated from dividing by the product of the velocity squared and the density, rather than by the dynamic pressure. Still other coefficients are odd multiples of the standard coefficients, but these are not absolute and will be abolished sooner or later.

In addition there are used the ratio of the lift to the drag, and the center of pressure. This latter is the point at the wing chord with respect to which the moment of the resultant air force is zero. This is generally a different point for each angle of attack, the center of pressure is said to "travel." It is expressed by its distance from the leading edge in per cent of the chord length.

Normal force and tangential force signify the components of the resultant air force at right angles to, and parallel to, the wing chord. The tangential force may become negative, whereas the drag is always positive. Their coefficients are formed in the standard way, and are denoted by C_n and C_t.

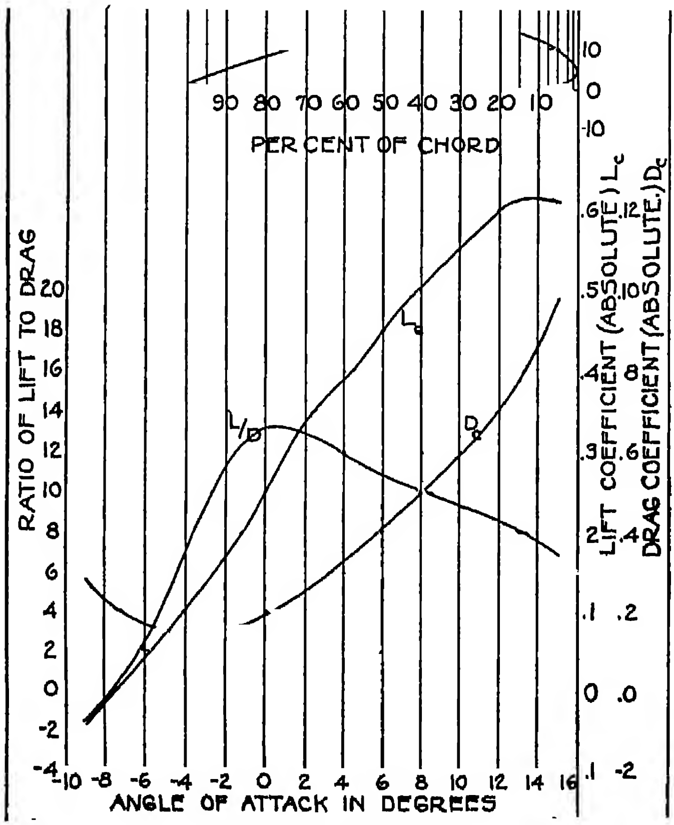


Figure 32. Lift Diagram

Proceeding now to the methods of plotting these coefficients, two different diagrams are chiefly used. The one diagram developed from the way the tests were made; the angle of attack of the model is varied and the air forces are measured each time. Accordingly, these forces, or rather their coefficients, are plotted against the angle of attack. This diagram then contains three curves, the lift coefficient, the drag coefficient and the center of pressure plotted against the angle of attack. The polar diagram or lift curve, on the other hand, has the lift coefficient

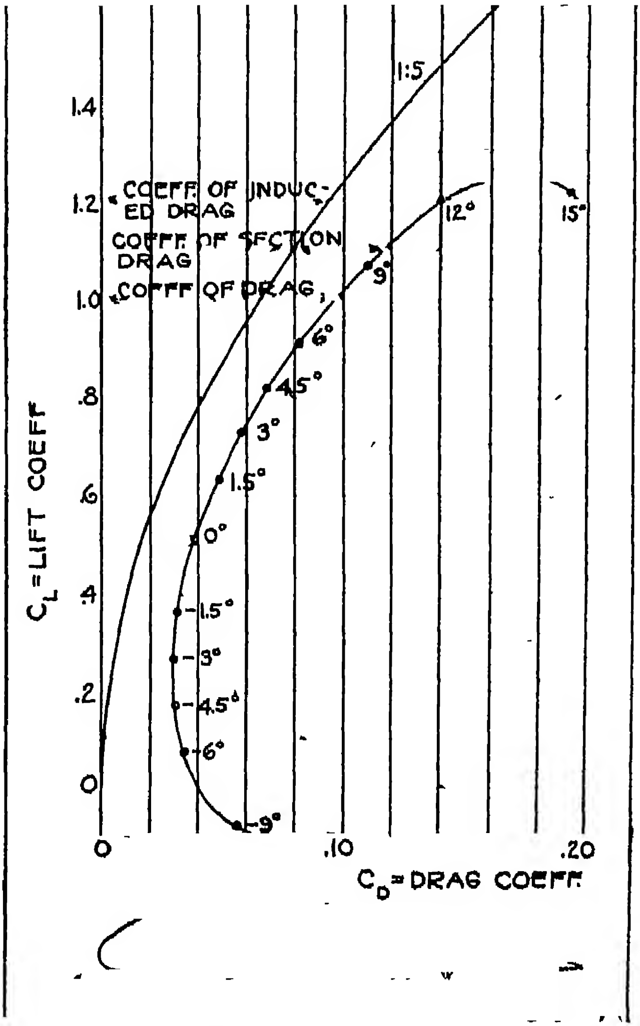


Figure 33. Polar Diagram

as the independent variable, and against it are plotted two curves, the drag coefficient and the moment coefficient. Lift and drag are plotted in their natural position, therefore the moment coefficient is plotted to the right, which is a somewhat unusual way. The angle of attack is usually inserted in writing at different points of the lift curve. The scale of the lift coefficient is one-fifth of the scale of the drag coefficient.

The polar curve more easily admits a comparison of results from tests made with models of different aspect ratio than the other diagram. We shall immediately see that the curve of the induced drag can be plotted into the polar diagram without any regard to the test results. This curve is a parabola that depends on the aspect ratio only. It is usually inserted and shows then at a glance the friction drag of the section. Furthermore, the moment curve in this diagram is a very regular curve making it easy to interpolate points between the tests. The center of pressure on the other hand is directly used by the designer. It seems as if both kinds of diagrams are going to be used side by side, the polar for research and the ordinary diagram for design.

The full advantage of the coefficients can only be derived if the theoretical relations between them are known. They are obtained by dividing the air forces in the standard way. The following equations appear then:

The induced angle of attack is

$$\alpha_i = C_L S/\pi b^2 \text{ in radians}$$

The induced drag:

$$C_{D_i} = C_L^2 S/\pi b^2$$

The lift coefficient as a function of the effective angle of attack in radians:

$$C_L = 2\pi\alpha_e$$

For biplanes, the aspect ratio is the ratio of the entire area of both wings divided by the square of the spa— The effective

aspect ratio, to be used in the computation of the induced angle of attack and drag, is the ratio of the area to the square of the effective span, discussed in Section 48. It is about 10% larger than the actual aspect ratio. Figures 32 and 33 show the same test results, plotted in the two kinds of diagrams.

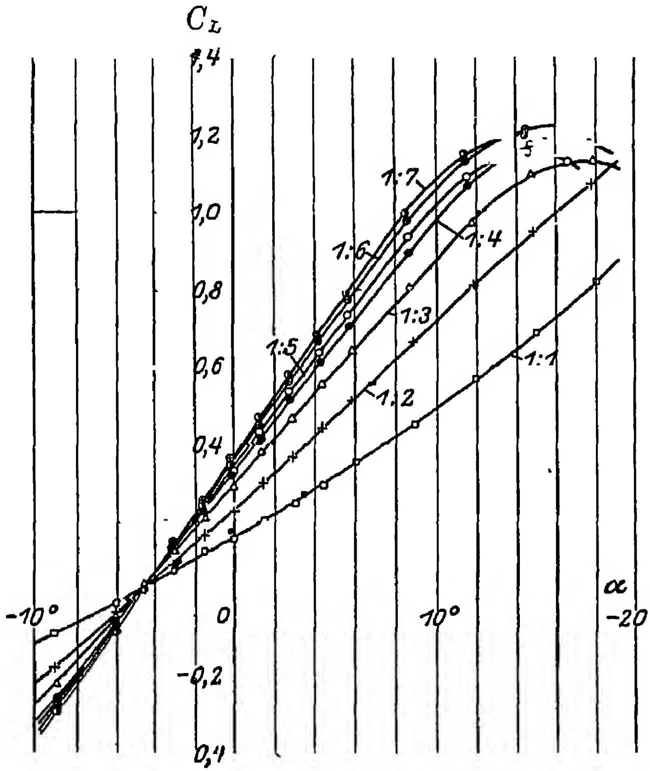


Figure 34. Lift Curves for Different Aspect Ratios

Numerous other polar diagrams published show the friction drag to be comparatively small within the useful range of the angle of attack and by far less variable than the total drag. The drag coefficient of any wing is computed from a model test by assuming the friction drag coefficient to be independent of

the aspect ratio. The induced drag is computed for the tested wing and for the designed wing; their difference is added or subtracted to the total drag observed. This becomes clearer in the diagrams. In the polar curve, the parabola of induced

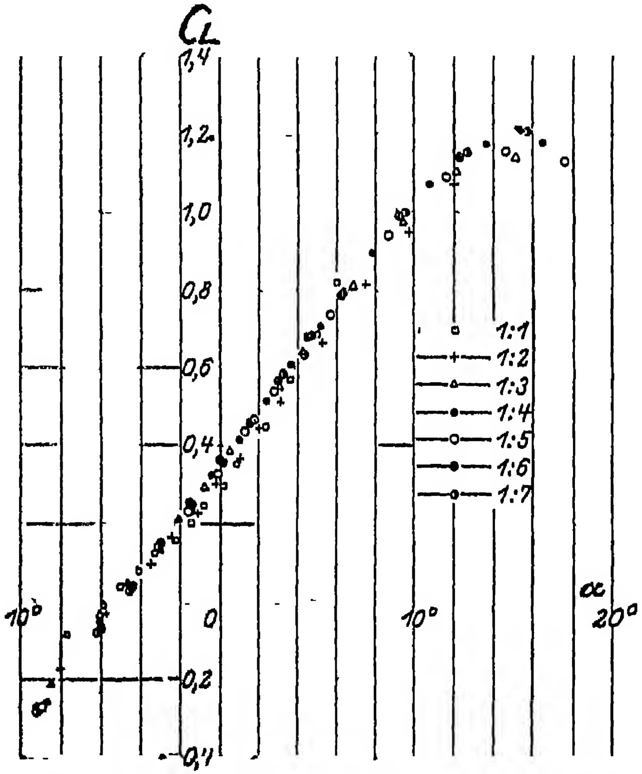


Figure 35. Reduced Lift Curves for Different Aspect Ratios

drag for the aspect ratio of the designed wing is plotted, and the friction drag, that is, the horizontal distance between the original parabola and lift curve, is added to the right. This has been done for five model tests represented in Figures 34 and 36. In Figure 36 the polar curves are plotted as observed, and in Figure 35 all are reduced to the same aspect ratio. It is indeed

remarkable how well they coincide afterwards with the curve obtained at that aspect ratio.

The same procedure can be followed with the angle of attack. Deduct the induced angle of attack of the model and add the induced angle of attack of the design.

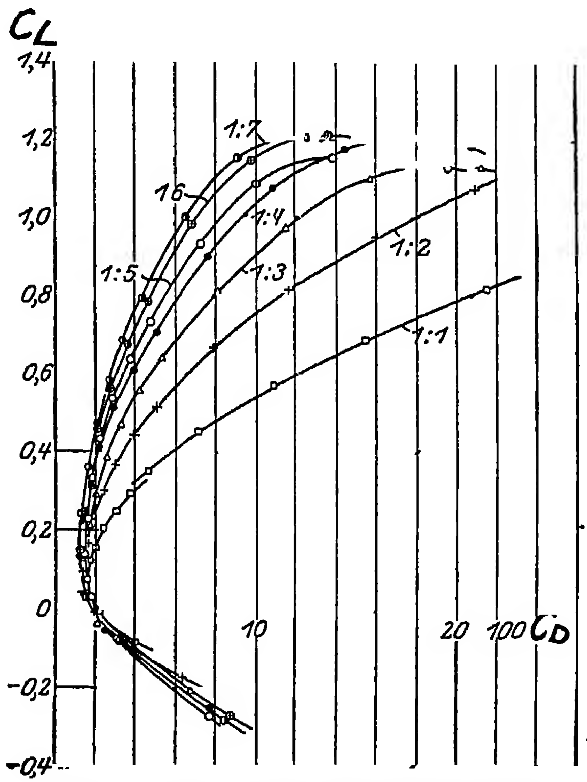


Figure 36. Polar Curves for Different Aspect Ratios

As the angle of attack increases more and more, the lift curve ceases to follow parallel to the theoretical curve, and breaks off. With some sections, it falls off, but with good sections the lift coefficients retain their maximum value over a large range of the angle of attack. The critical angle at which the lift ceases to

increase is called the burble point, the air flow becomes burbling and the wing leaves a wake of whirls behind it. The drag is correspondingly large. With sections having equal upper and lower curves this burbling point is at a value of about 0.8 for

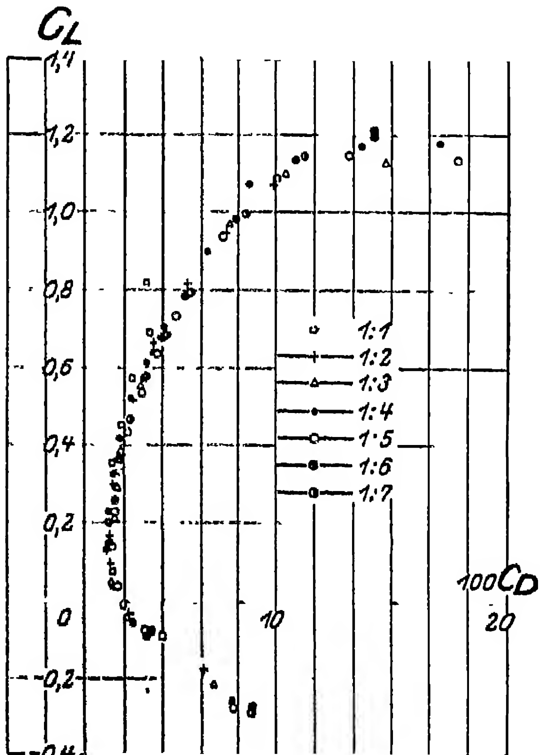


Figure 37. Reduced Polar Curves for Different Aspect Ratios

the lift coefficient. S-shaped sections may reach a lift coefficient up to 1.2 with equal absence of the travel of the center of pressure as with symmetrical sections, and with as small a drag. Curved sections may reach a lift coefficient up to 1.8.

The minimum friction drag coefficient is about 0.01 up to considerable values of the lift coefficient. This is of the same

order of magnitude as the surface friction, and a good confirmation of the theory.

In Figure 38 is represented the polar curve of an S-shaped section so designed that the moment is zero at all angles of attack.

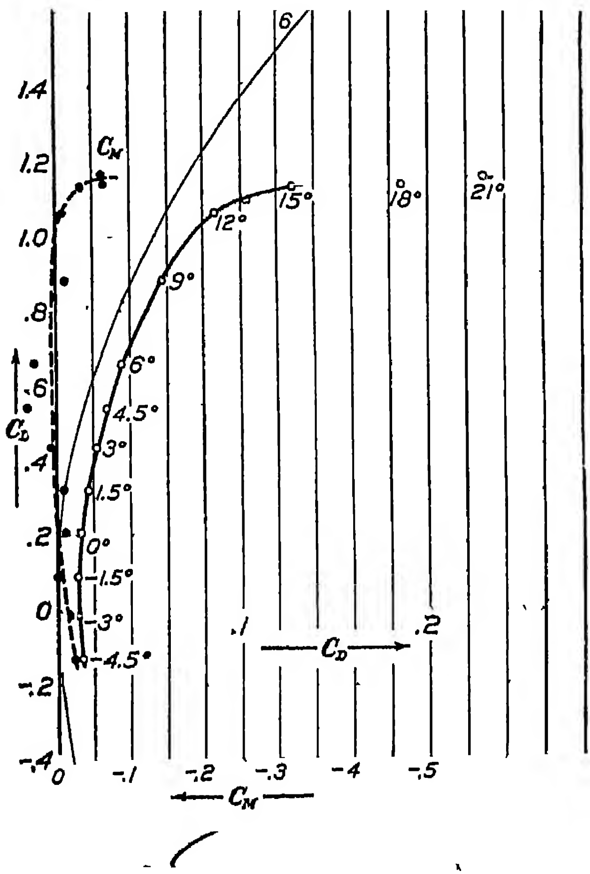


Figure 38. Polar Curve and Moment Curve of an S-shape Wing Section

This is, of course, the moment with respect to 25% of the chord. The diagram shows very good agreement with the prediction; the moment curve is practically coinciding with the vertical axis.

The theoretical relation between the lift coefficient and the effective angle of attack is less exactly confirmed by experiments. The lift is smaller than expected. Just how much could only be decided by statistics of the tests. No such statistics have been made for the last ten years. In 1918 the author collected statistics of the wing section tests then available. The rise of the lift coefficient per degree increase of the effective angle of attack was found to be around 0.10. This commits itself easily

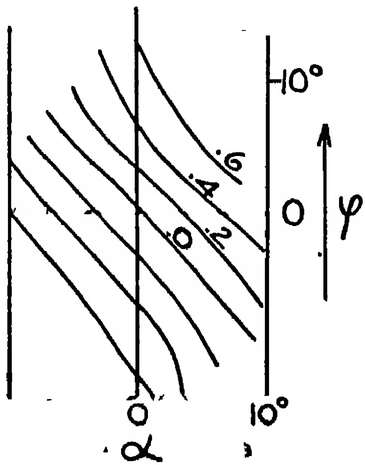


Figure 39. Plot of Lift Produced by Elevator Displacement

to memory. The theoretical value is 2/57.3. The lift would then be 90% of the theoretical value. This is partially explained by the finite thickness of the section, particularly the curvature of the leading edge, reducing the effective chord length. Even without that the agreement is not so bad. Furthermore, the tests then available were made at wind speeds of the tunnel now considered as unusually small. The more recent tests may give a larger value of this very important physical constant.

In Figure 39 the lift observed with a stabilizer and elevator model at differing settings of the elevator and at different angles

of attack of the stabilizer is plotted. The resulting curves are practically parallel straight lines and confirm the linear relation suggested by theory. The numerical relation between the angle of setting of the elevator and the lift coefficient is likewise as predicted within reasonable limits.

68. Wing Moments

Test results referring to the relation between the induced yawing moment and the rolling moment of wings are represented in Figure 40. The coefficient of yawing moment is plotted against the coefficient of rolling moment. Both coefficients are obtained by dividing the moments by the same quantity, which is the product of the dynamic pressure, the square of the span, and the sum of the chords of the upper and lower wings.

A glance at this diagram makes it at once apparent that the type of relation suggested by the theory has actually been observed by these tests. The individual curves for particular angles of attack are substantially straight lines, passing through the point of origin, which shows that the ratio of the two moments is constant for each angle of attack. Furthermore, the tangents of the slopes of these straight lines are substantially a linear function of the angle of attack. The slope zero occurs at an angle of attack about $-2°$. This angle $-2°$ is the angle of attack of zero lift for the airplane in question. The yawing moment appears to be proportional to the lift coefficient, in agreement with the formula. It remains only to examine how far the agreement includes the magnitude of the factor $3/\pi$. Since the type of relation has been demonstrated to agree, it is sufficient to check the factor for one condition only.

We choose for such check the angle of attack 12°, having a lift coefficient $C_L = .96$. The diagram gives $M_y/M_r = .25$. The wing area of the airplane is 188 sq. ft., and the span is 26.6 ft., giving an aspect ratio

$$b^2/S = 26.6^2/188 = 3.75$$

Hence our formula gives likewise:

$$M_y/M_r = 3/\pi \cdot C_L = .96 \times 3/3.75\pi = .25$$

That happens to be a very good agreement. An error of some per cent should be expected, as, in the computation, the lift distribution is somewhat arbitrarily assumed. Substituting the effective aspect ratio of the biplane for the nominal aspect ratio b^2/S would indeed decrease the result somewhat.

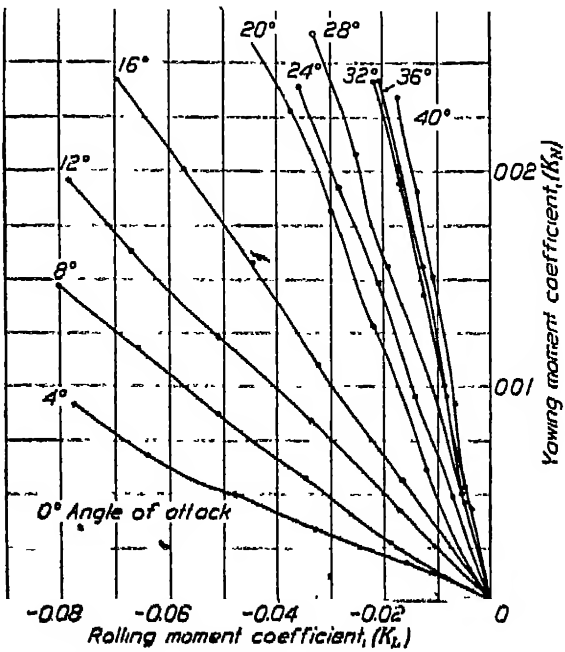


Figure 40. Plot of Rolling and Yawing Moment

69. Propeller Forces

Propeller research has been carried on less extensively than wing research and has remained more superficial. No standard practice about the use of the coefficients or of diagrams has developed. The slipstream theory suggests the use of the thrust coefficient

$$C_T = \frac{T}{V^2\rho/2 \cdot D^2\pi/4}$$

giving the slipstream velocity v at the propeller by

$$v/V = \tfrac{1}{2}(\sqrt{1 + C_T} - 1)$$

The progress of the science would be best served by plotting this ratio, computed from the observed thrust coefficient, against the ratio U/V of the tip velocity divided by the velocity of flight. The torque or power coefficient corresponding to the thrust coefficient given would be

$$C_P = \frac{P}{V^3 \rho/2 \cdot D^2 \pi/4}$$

Figures 41 and 42 give the relations between the two coefficients and the slipstream velocity ratio graphically. The equation between the torque coefficient and the slip ratio w/V is

$$C_P = \tfrac{1}{2}(w/V)^3 + 2(w/V)^2 + 2(w/V)$$

The available tests show the minimum losses to be in substantial agreement with the slipstream theory together with the friction losses of ordinary wings. There is, therefore, not much possibility left for a considerable improvement of the maximum propeller efficiency. The development is chiefly directed towards improving the overall efficiency over a larger range of conditions.

The computation of the thrust is not as important right now as it was formerly, since adjustable pitch propellers are now much in use. The comparison of the observed thrust with the computed one is still of interest, as it throws some light on the conditions of flow.

Figure 43 contains three slip curves, taken from an analysis of propeller model tests made by W. F. Durand. Computation gives a mean slope of the slip curves .10, the tests give .13. The explanation of this large difference is beyond the scope of this book, nor has it yet found its way into literature.

70. Parasite Drag

The theory of the drag of blunt bodies is very scanty. All that can be said is that the average wake velocity will not greatly

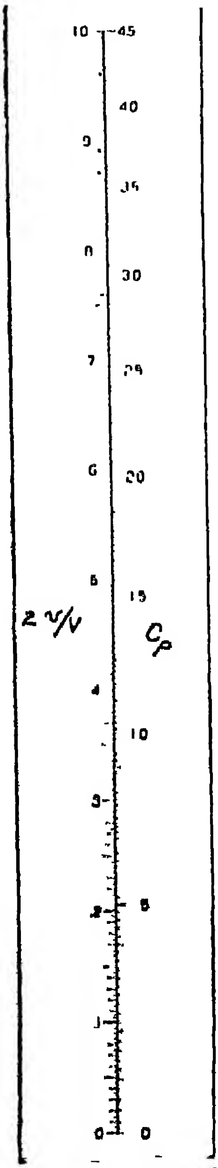


Figure 41. Abacus for Thrust Coefficient and Relative Slip Velocity

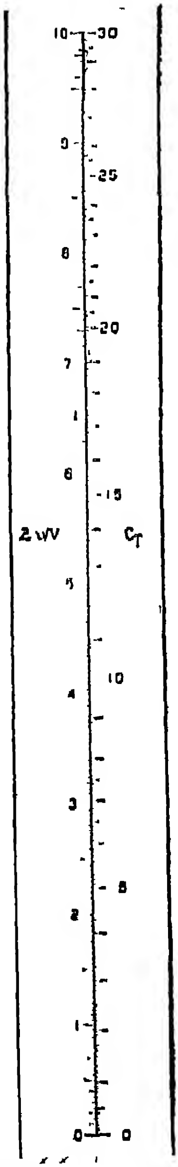


Figure 42. Abacus for Torque Coefficient and Relative Slip Velocity

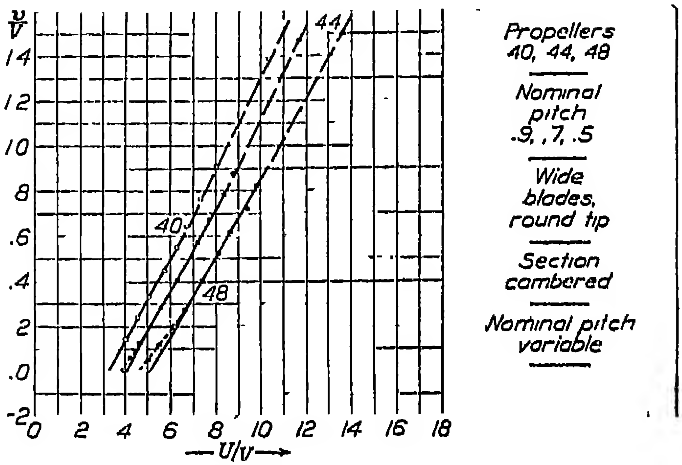


Figure 43. Plot of Relative Slip Velocity

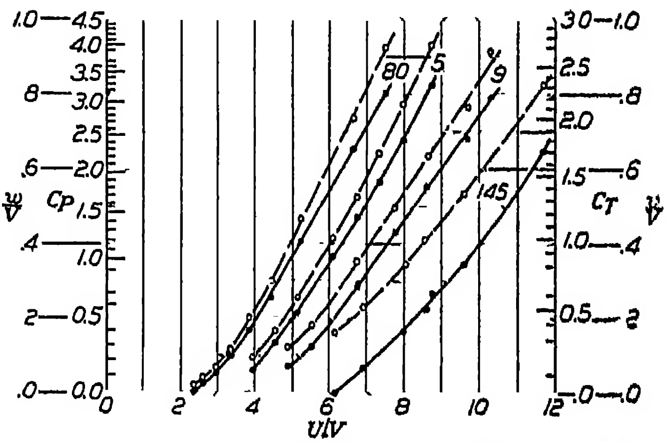


Figure 44. Relative Slip Velocity Computed from Observed Thrust and Torque

exceed the velocity of motion, nor will its cross-section greatly exceed the front projection of the body. This would give a drag coefficient referring to this projection near 1. Experience shows it to be up to 1.7. This includes circular rods and cables. Nor does it seem possible to obtain a larger coefficient as with para chutes where it would be desired.

The drag coefficients can be diminished by streamlining. Strut sections have a drag coefficient down to .10 or .09. The ratio of length to width is then 2.5 to 3. Longer sections have a larger drag coefficient again as the surface friction assumes then too large values. Streamlined spindles may have a drag coefficient down to .04.

71. Synopsis

The drag coefficient of elongated airships of the rigid type goes down to .02, where the coefficient refers to the ⅔ power of the volume. The coefficients for blunter shapes are smaller, down to .01. The wing forces are represented by the lift coefficient C_L and the drag coefficient C_D referring to the wing area. The minimum drag coefficient is .01, the maximum lift coefficient up to 1.8. The induced angle of attack is $C_L/\pi \cdot S/b^2$; the induced drag coefficient is $C_L{}^2/\pi \cdot S/b^2$. Hence, it appears as a parabola in the polar curve lift coefficient plotted against the drag coefficient. The theoretical maximum efficiency of a propeller is

$$\eta = \frac{2}{1 + \sqrt{1 + C_T}}$$

where C_T is the thrust coefficient referring to the velocity of flight and to the area of the propeller disc. The lift coefficient per degree effective angle of attack is about .1.

The smallest drag coefficient of spindles is .04, of struts is .09, the drag coefficient of blunt shapes is 1.0 to 1.7 including cables and flat discs.

72. Problems and Suggestions

1. The drag of a strut 2 in. in width and 6 ft. in length at a velocity of 220 ft./sec. and air density of 1/420 lb. sec^2/ft^4 is 11 lb. Compute its drag coefficient.

2. The drag per foot length of a strut is .6 lb. How large is the drag of a strut twice that width with a similar cross-section, all other conditions remaining equal?

3. The horsepower absorbed by an airplane flying at the same altitude with different velocity is proportional to $C_D/C_L^{3/2}$.

4. The wing loading of an airplane (weight per sq. ft. wing area) is 6 lb./sq. ft., and the plane is flying at a velocity of 100 mi./hr. in air of the standard density at sea level 1/420 lb. sec^2/ft^4. At which lift coefficient is it flying?

5. The center of pressure of a wing lies at a distance C_m/C_n from its leading edge.

6. The drag coefficient of a wing model with an aspect ratio 6 at the lift coefficient 0.8 is 0.1. How large is it for the aspect ratio 4 and the same wing section and lift coefficient?

7. A circular disc with the drag coefficient 1.0 has the same drag as a round cigar-shaped spindle with the drag coefficient 0.07. How large is the maximum diameter of the spindle?

8. How large is the lowest landing velocity of an airplane weighing 2,000 lb., the wing area being 400 sq. ft., the air density 1/420 lb. sec^2/ft^4, and the maximum lift coefficient of the wing section 1.4?

9. The coefficient of induced drag of an airplane is inverse to the fourth power of its velocity.

10. Compute the maximum theoretical efficiency of a propeller, if its thrust coefficient is equal to 1.

73. Conclusion

The application of fluid dynamics to aeronautic problems is as simple as the application of any other technical science. The results are as useful and indispensable for the further progress of the art. This application requires mere arithmetic. The

underlying principles of the science are not more involved than the principles of the theory of elasticity or of the electro-magnetic theory; they appear more difficult only because the science is newer and there are not yet many competent teachers. This difficulty will be removed in time, and in the same degree more good will be derived from the science.

Theory fails thus far to give sufficient account of the causes of turbulence and whirling. It is this question around which further progress of the science centers, and important progress is being made from year to year in the European centers of research. Nowhere, however, is fundamental flow research carried on at present in this country. As the results of this European work, theory will become even more useful and will contribute even more towards improvement of aeronautic arts.

APPENDIX

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NOMENCLATURE

The following aeronautical expressions are used with a meaning not defined elsewhere:

Angle of attack. Actual or geometric angle between a wing section chord and its motion relative to the air unaffected by the wing.

Angle of yaw. The angle between the wing span and the plane at right angles to its motion relative to the air.

Aspect ratio. The ratio of the wing span and the average wing chord, obtained by dividing the square of the span by the entire wing area of the airplane.

Bow. The front portion of an airship hull.

Drag. The air force component parallel to the motion of the solid relative to the air unaffected by it.

Elevator. In this book, any movable surface hinged behind an immovable surface and drawn through the air. In practice, only the horizontal tail control surface is called elevator.

Elevator section. The cross-section through the elevator and the immovable surface in front of it.

Gap. The distance of one wing in a biplane or multiplane from the chord of the next.

Lift. The air force component at right angles to the motion of the solid relative to the air unaffected by it.

Strut. Structural compression member drawn through the air.

Wind tunnel. An air jet to which objects will be exposed in order to study their air pressures and forces.

Wing. A surface moved through air for the creation of lift (not half such surface).

SOLUTIONS TO PROBLEMS

Chapter I

Pr. 1. $\Phi = \dfrac{-A}{\sqrt{x^2+y^2+z^2}}, \quad \dfrac{\partial\Phi}{\partial x} = \dfrac{+Ax}{(x^2+y^2+z^2)^{3/2}},$

$$\frac{\partial^2\Phi}{\partial x^2} = \frac{-3Ax^2}{(x^2+y^2+z^2)^{5/2}} + \frac{A}{(x^2+y^2+z^2)^{3/2}}$$

Similarly $\dfrac{\partial^2\Phi}{\partial y^2} = \dfrac{-3Ay^2}{(x^2+y^2+z^2)^{5/2}} + \dfrac{A}{(x^2+y^2+z^2)^{3/2}}$

and $\dfrac{\partial^2\Phi}{\partial z^2} = \dfrac{-3Az^2}{(x^2+y^2+z^2)^{5/2}} + \dfrac{A}{(x^2+y^2+z^2)^{3/2}}$

and thus $\partial^2\Phi/\partial x^2 + \partial^2\Phi/\partial y^2 + \partial^2\Phi/\partial z^2 = 0$, which is Laplace's Equation.

$$u = \partial\Phi/\partial x = A/x^2, \text{ for } y = 0, z = 0$$

$$V = \partial\Phi/\partial y = 0, \text{ and } w = \partial\Phi/\partial z = 0, \text{ for } y = 0, z = 0$$

Hence the velocity along the x axis is A/x^2, directed away from the origin.

Pr. 2. $\dfrac{\partial^2\Phi}{\partial x^2} = \dfrac{\partial u}{\partial x} = \dfrac{-2Axy}{(x^2+y^2)^2}, \quad \dfrac{\partial^2\Phi}{\partial y^2} = \dfrac{\partial v}{\partial y} = \dfrac{2Axy}{(x^2+y^2)^2}$

The flow is one of constant density, since $\partial^2\Phi/\partial x^2 + \partial^2\Phi/\partial y^2 = 0$

$$u_y - v_x = \frac{A}{x^2+y^2} - \frac{2Ay^2}{(x^2+y^2)^2} + \frac{A}{x^2+y^2} - \frac{2Ax^2}{(x^2+y^2)^2} = 0,$$

$\therefore$ zero rotation.

Pr. 3. $u = \partial\Phi/\partial x = A,\ v = B,\ w = C, \quad V = \sqrt{u^2+v^2+w^2}$
$= \sqrt{A^2+B^2+C^2}$

Pr. 4. Given $\Phi_{xx} + \Phi_{yy} + \Phi_{zz} = 0$; take the x derivative of this function: it is $\Phi_{xxx} + \Phi_{yyx} + \Phi_{zzx} = 0$, but by the commutative law $\Phi_{xxx} + \Phi_{xyy} + \Phi_{xzz} = 0$, hence Φ_x satisfies Laplace's Equation, Q. E. D.

Pr. 5. This is a corollary of the theorem proved in Problem 4, and follows directly by the definition of velocity potential.

Pr. 6. It will be seen that the flow is parallel to the x axis on all the axes, with $V = A/r^3$. Elsewhere the flow is oblique.

Pr. 7. $v = 0$ and $w = 0$ when $x = 0$, hence parallel flow.
$\frac{B}{A} = \frac{1}{x^2 + y^2 + z^2}$ and $ux + vy + wz = 0$, showing that the flow is normal to the radius of the sphere at its surface.

Pr. 8. $V^2\,\rho/2 = 120^2/840 = 17.1$ lb./ft.^{2}

Pr. 9. $V = u = \frac{A}{y^2 + z^2} + B = 2B$

Pr. 10. $60^2/840 = 4.3$ lb./ft.^{2} dynamic pressure at a large distance.
$120^2/840 = 17.1$ lb./ft.^{2} dynamic pressure at surface of sphere.
$17.1 - 4.3 = 12.8$ lb./ft.^{2} below distant pressure.

Pr. 11. The radial component of the velocity of motion of the cylinder must equal the radial component of velocity of flow. Introducing the polar coordinates r and θ, where $r^2 = x^2 + y^2$ and $\tan\theta = y/x$, the former velocity becomes $\cos\theta$; the latter $= d\Phi/dr = d/dr \cdot (-\cos\theta/r) = \cos\theta/r^2 = \cos\theta = x$, Q. E. D. In a portion of the cylinder of length $= l$, the kinetic energy $= T = \rho/2 \cdot \int_0^{2\pi} \cos^2\theta d\theta$ $= \rho\pi/2$, and thus the volume of apparent additional mass is π.

Pr. 12. $m' = 5/4 \times m = 5/4 \times W/g$; $a = W/m' = 4/5 \times g$.

Chapter II

Pr. 1. $x^2/a^2 + y^2/b^2 = 1$, equation of ellipse.
$S = \pi y^2 = \pi b^2 - (\pi b^2/a^2)x^2$.
$F = k \cdot dS/dx = 2\pi\, k \cdot b^2/a^2 \cdot x = k'x$, Q. E. D.

Pr. 2. $M =$ Vol. $(k_2 - k_1)q \sin 2\alpha$
$= 4\pi/3 \times 35^2 \times 350 \times .940 \times 90^2/840 \times .139 = 2{,}260{,}000$ lb. ft.

Pr. 3. $dF/dx = V^2\,\rho/2 \cdot \sin 2\alpha(k_2 - k_1)dS/dx$.
$y^2 = 35^2 - (35^2/350^2)x^2$; $x = 50, y^2 = 1{,}200$; $x = 150, y^2 = 1{,}000$
$F = 90^2/840 \times .139 \times .940 \times 200\pi = 791$ lb.

Pr. 4. $\dfrac{dy}{dx} = -\dfrac{b}{a}\dfrac{x}{\sqrt{a^2 - x^2}} = .0298$

$V_1 = 90 + 90 \times .139 \times .0298 = 90 + .4 = 90.4$, $q_1 = 9.70$. lb./ft.^{2}

$V_2 = 90 - .4 = 89.6$, $q_2 = 9.55$ lb./ft^2, $q_1 - q_2 = 0.15$ lb./ft.^{2}

Alternate method, by force on 1 ft. segment, assuming uniform pressure gradient dp/dy:

$\dfrac{dF}{dx} = q \sin 2\alpha(k_2 - k_1)\dfrac{dS}{dx} = 9.65 \times .139 \times .94 \times$

$\dfrac{2\pi}{100} \times 100 = 7.9$ lb./ft.

$\dfrac{dp}{dy} = \dfrac{dF/dx}{S} = \dfrac{7.9}{1{,}125\pi}$, $p = 2y\dfrac{dp}{dy} = 2\sqrt{1{,}125}\,\dfrac{7.9}{1{,}125\pi}$

$= 0.15$ lb./ft.^{2}

Pr. 5. $2{,}260{,}000/275 = 8{,}220$ lb.

Pr. 6. $V^2\rho/2 = 9.65$ lb./ft.^{2} above atmospheric.

Pr. 7. $q = \dfrac{90^2 + 12.6^2}{840} = 9.84$ lb./ft.^{2}, 0.19 lb./ft.^{2} below atmospheric.

Pr. 8. $q = \dfrac{(1.021 \times 90)^2 + (1.96 \times 6.3)^2}{840} = 10.22$ lb./ft.^{2}, 0.57 lb./ft.^{2} below atmospheric.

Pr. 9. $\dfrac{dy}{dx} = \dfrac{1.021 \times 90}{1.96 \times 6.3} = 7.53$; $\dfrac{x}{\sqrt{350^2 - x^2}} = 7.53$; $x = \dfrac{7.53}{\sqrt{57.7}}$ $\times 350 = 347$, highest pressure is 3 ft. from nose on bottom of bow.

$\dfrac{dy}{dx} = \dfrac{1}{7.53} = .133$, $x = \dfrac{.133}{\sqrt{1.0176}}\,350 = 46.5$ ft., distance of point of lowest pressure from center.

Pr. 10. $.021 \times (4/3)\pi \times 35^2 \times 350 \times (90^2/840) = 376{,}000$ ft. lb.

Pr. 12. $F = \dfrac{V^2}{r}\rho \cdot \text{Vol.} = \dfrac{90^2 \times (4/3)\pi \times 35^2 \times 350}{1{,}000 \times 420} = 3{,}470$ lb.

(= centrifugal force)

Pr. 11. $M = Fa = 3{,}470 \times 275 = 954{,}000$ lb. ft.

Pr. 13. $y = cx$, $S = \pi y^2 = \pi c^2 x^2$; $F = k\,dS/dx = 2k\pi c^2 x = k'x$, Q.E.D.

Chapter III

Pr. 1. $v = A\cos\delta/\sin\delta$; $A = \frac{1}{2}$, $z = \frac{1}{2}$, $\cos\delta = \frac{1}{2}$, $\sin\delta = .866$;
$v = .289$ ft./sec.

Pr. 2. $v = \dfrac{\cos\delta}{\tan\delta} - \sin\delta = \dfrac{\cos^2\delta - \sin^2\delta}{\sin\delta} = \dfrac{\cos 2\delta}{\sin\delta}$
$v\sin\delta = A_1\cos\delta + 2A_2\cos 2\delta + 3A_3\cos 3\delta \ldots = \cos 2\delta$,
thus $A_2 = \frac{1}{2}$, $A_1 = A_3 = 0$
$u\sin\delta = A_1\sin\delta + 2A_2\sin 2\delta + 3A_3\sin 3\delta \ldots = \sin 2\delta$;
$u = \sin 2\delta/\sin\delta$
$\Phi = A_1\sin\delta + A_2\sin 2\delta + A_3\sin 3\delta + \ldots = \frac{1}{2}\sin 2\delta$

Pr. 3. Yes, by definition of Φ.

Pr. 4. $u = \cos\delta$; $u\sin\delta = A_1\sin\delta + 2A_2\sin 2\delta + 3A_3\sin 3\delta + \ldots$
$= \frac{1}{2}\sin 2\delta$; $v\sin\delta = \frac{1}{2}\cos 2\delta$; $v = \cos 2\delta/2\sin\delta$

Pr. 5. $v = A\cos\delta/\sin\delta$; $A = 30$, $\cos\delta = .4$, $\sin\delta = .3$, $v_1 = 40$ ft./sec.
$A = 30$, $\cos\delta = 0$, $\sin\delta = 1$, $v_2 = 0$
$p = v_1^2\rho/2 = 40^2/840 = 1.91$ lb./ft^2

Pr. 6. $u = Kz = K\cos\delta$; $z = 1$, $u = 30$, $\therefore K = 30$ ft./sec.

Pr. 7. Edge: $\cos\delta = 1$, $\cos 2\delta = 0$, $\sin\delta = 0$

$$v = \frac{K\cos 2\delta}{2\sin\delta} = \frac{K}{2}\frac{(1 - 2\sin^2\delta)}{\sin\delta} = \frac{K}{2}\left(\frac{1}{\sin\delta} - 2\sin\delta\right)$$

$$= 15\left(\frac{1}{\sin\delta}\right)_{\delta = 0}$$

Pr. 8. $u = 15$, $v = 15\cos\delta/\sin\delta = 15\,(1/\sin\delta)$ at end.

Pr. 9. One-fourth from end.

Pr. 10. $T = A_n^2 n\pi\,\rho/2$, $n = 2$, $A_2 = K/4 = 7.5$, $T = 56.25\,\pi\rho$
$I = \dfrac{2T}{\omega^2}$; Angular vel. $= \omega = \dfrac{30}{1}$ rad./sec.; $I = \dfrac{2A^2\pi\rho}{(4A)^2} = \dfrac{\pi\rho}{8}$

Pr. 11. $\Phi = \log z$; $F = \dfrac{d\Phi}{dz} = \dfrac{1}{z} = \dfrac{1}{x + iy} = \dfrac{x}{x^2 - y^2} + i\dfrac{y}{x^2 - y^2}$;

Longitudinal vel. $= \dfrac{1}{x}$, transverse zero, along the line.

Pr. 12. $\Phi = i \log z; F = \frac{i}{z} = \frac{-y}{x^2 - y^2} + i\frac{x}{x^2 - y^2}$

Longitudinal velocity zero, transverse $= -1/x$, along the line.

Pr. 13. $\Phi_1 = \sin\delta; v_1 = \cos\delta/\sin\delta = 1/\sin\delta_{(\delta=0)}$
$v_2 = A/\sin\delta; v_1 = v_2, A = 1; \Phi_2 = \delta$

Chapter IV

Pr. 1. $L = qS \cdot 2\pi(\alpha + .16); M_c = qS\pi\alpha$
$M_c/L = 1; \alpha = 2\alpha + .16; \alpha = -.16 = -9.2°$

Pr. 2. $C_L = qS \times 2\pi \times .16/qS = 1.0$

Pr. 3. From Figure 11, for $e/c = .33$, $\alpha_{eq}/\Phi = .66$; $\Phi = 5/.66 = 7.5°$

Pr. 4. Negative, as seen from the graphical method.

Pr. 5. Lift by Gauss' method, $n = 2$:
11% chord: $x = .78$, $\xi = .05 \times \sqrt{1 - x^2} = .0244$
89% chord: $\xi = -.0244$
$\alpha_0 = 265(-.0244)/2 + 32(.0244)/2 = -2.84°$
$5 - 2.84 = 2.16° = .0395$ radians; $C_L = 2\pi\alpha = .247$

Pr. 6. The graphical method gives 2.8°

Pr. 9. $L = V^2(\rho/2)S \cdot 2\pi\alpha = 30^2/16 \times .9 \times 6.28 \times .105 = 33.4$ kg./m.

Pr. 10. Finite velocity at both ends requires a symmetrical flow of zero lift; i.e., no circulation flow. From the solution of Problem 5 we see that this condition occurs at 2.84°.

Pr. 11. Down.

Pr. 12. α is halved and S doubled, $L = qS2\pi\alpha$ = constant.

Pr. 13. $2\pi\omega l$

Pr. 14. This fact may be seen by a consideration of the velocity distribution of the transverse and circulation flows—the velocities add at the stabilizer and subtract at the elevator.

Chapter V

Pr. 1. $\alpha_i = \dfrac{L}{\pi b^2 V^2 \rho/2} = \dfrac{1{,}200 \times 840}{3.14 \times 30^2 \times 100^2} = .0356 \text{ rad.} = 2.0°$

Pr. 2. $D_i = L\alpha_i = 43$ lb.

Pr. 3. $b' = \sqrt{b^2 + 4bh/\pi} = \sqrt{900 + \dfrac{4 \cdot 30 \cdot 5}{3.14}} = 33$

$\alpha = \dfrac{2{,}400 \times 840}{3.14 \times 33^2 \times 100^2} = .0588 \text{ rad.} = 3.4°$

$D_i = 2{,}400 \times .0588 = 142$ lb.

Pr. 4. $L = \dfrac{qS \cdot 2\pi\alpha}{1 + 2S/b^2} = \dfrac{3}{4} \times \dfrac{100^2}{840} \times 150 \times 6.28 \times .0348 = 290$ lb.

Pr. 5. $\alpha = \dfrac{L(1 + 2S/b^2)}{qS \cdot 2\pi} = \dfrac{1{,}200(1 + 10/25)840}{100^2 \times 125 \times 6.28} = .180$ rad.

$= 10.3°$ final.

$\alpha_0 = \dfrac{1{,}200(1 + 10/30)840}{100^2 \times 150 \times 0.28} = .143 = 8.2°$ initial.

$10.3 - 8.2 = 2.1°$ increase.

Pr. 6. α_0 is increased by 25% or 2.1°.

Pr. 7. Both increased 25%; 2.5°, 54 lb.

Pr. 8. From Figure 11, when $e/c = .25$, $\alpha_{eq}/\Phi = .60$; $\alpha_{eq} = .6 \times 3 = 1.8°$.

$\alpha' = 1.8 + 1 = 2.8° = .049$ rad.

$L = \dfrac{qS \cdot 2\pi\alpha'}{1 + 2S/b^2} = \dfrac{5}{9} \times \dfrac{100^2}{840} \times 40 \times 6.28 \times .049 = 82$ lb.

Pr. 9. $\alpha = 4°$ (stabilizer) $+ 1.8°$ (elevator) $- 4°$ (downwash) $= 1.8°$

$L = 82 \times 1.8/2.8 = 52.7$ lb.

Pr. 10. $1 + 2S/b^2 = 9/5 \times .9$; $b = 11.4$ ft.

Pr. 11. $\dfrac{14.5}{17} \times \dfrac{1 + 4/17}{1 + 4/14.5} = .826$

Pr. 12. $\alpha = \dfrac{L(1 + 2S/b^2)}{qS \cdot 2\pi} = \dfrac{1{,}400(1 + 500/36^2)}{120^2 \times .001 \times 250 \times 6.28} = .086$ rad.

$= 4.9°$

$3 - 4.9 = -1.9°$

Chapter VI

Pr. 1. $1{,}600 \times 8/60 = 213$ ft./sec.

Pr. 2. $140 \times 60/8 = 1{,}050$ r.p.m.

Pr. 3. $v = \dfrac{S/D^2}{1 + S/D^2 \cdot D\pi/p}\left(U - V\dfrac{D\pi}{p}\right) = \dfrac{(10 \times 11)/(12 \times 10^2)}{1 + 11/120 \times 10\pi/8}$

$\left(\dfrac{1{,}600}{60} \times 10\pi - 140\,\dfrac{10\pi}{8}\right) = 19.5$ ft./sec.

Pr. 4. $T = 2D^2(\pi/4)(V + v)v\rho = 2 \times 100 \times .785 \times 160 \times 19.5/420$
$= 1{,}190$ lb.

Pr. 5. $N = \dfrac{TV}{1 - v/V} = \dfrac{1{,}190 \times 140}{1 - 19.5/140} = 194{,}000$ ft. lb./sec. $= 352$ HP.

Pr. 6. $T = 1{,}190/2 = 595$; $N = 352/2 = 176$ HP.

Pr. 7. $v = \dfrac{11/(12 \times 9)}{1 + 11/(12 \times 9) \times 9\pi/8}\left(\dfrac{1{,}800}{60} \times 9\pi - 140\,\dfrac{9\pi}{8}\right)$
$= 26.5$ ft./sec.
$T = 2 \times 81 \times .785 \times 166 \times 26.5/420 = 1{,}330$ lb.

Pr. 8. $v(V + v) = 19.5 \times 160 = 3{,}120$ originally;
$3{,}120 \times 1{,}750/1{,}190 = 4{,}590$ finally.
$v(140 + v) = 4{,}590$; $v = 27.4$ finally.
r.p.m./60 $-$ 140/8 = 9.2 originally; $9.2 \times 27.4/19.5 = 12.9$ finally.
r.p.m. $= 60(12.9 + 17.5) = 1{,}825$ r.p.m.

Pr. 9. $2v = \dfrac{2 \times 11/120}{1 + 11/120 \times 10\pi/8}\left(\dfrac{1{,}200}{60}\,10\pi - 60\,\dfrac{10\pi}{8}\right) = 53$ ft./sec.
$60 - 53 = 7$ ft./sec.

Pr. 10. $D^2(V + v)v =$ constant; $(V + v)v = 3{,}120$ originally;
$3{,}120 \times (3/5)^2 = 1{,}120$ finally; $v(140 + v) = 1{,}120$;
$v = 7.6$ finally.
$N = \dfrac{TV}{1 - v/V} = \dfrac{1{,}190 \times 140}{1 - 7.6/140} = 176{,}000 = 320$ HP., a saving of 32 HP., 9.1%.

Chapter VIII

Pr. 1. The drag per foot is doubled.

Pr. 2. Velocity $= 100\sqrt{2} = 141$ mi./hr.

Pr. 3. $\sqrt{2}/1$, since the drag is the same.

Pr. 4. $D'V' = DV;\ V'/V = D/D'$
$D' = D/2 \cdot (V'/V)^2 = D/2 \cdot D^2/D'^2$, or $D' = D/\sqrt[3]{2} = .794D$
Thus drag, and lift, must be reduced 20.6%.

Pr. 5. $P'/P = (11/8)^6 = 4.94$

Pr. 6. $R = Vl/\nu = \dfrac{90}{60} \times 88 \times 10 \times 50 \times 144 = 9{,}500{,}000$

Pr. 7. $\nu = \mu/\rho$; since μ is constant, ν varies as $1/\rho$

Pr. 8. Ten times normal, or 1/42 lb. sec.^{2}/ft.^{4}

Pr. 9. Five times normal, 1/84 lb. sec.^{2}/ft.^{4}

Pr. 10. Section 60 shows that since Reynolds Number is the same in the two cases, the pressures are proportional to the square of the velocity, and the air force varies as AV^2. Since R and ν are given constant, it appears that Vl, and hence AV^2, are constant; so the air force is constant.

Pr. 11. $R = Vl/\nu = 80 \times 10 \times 50 \times 144 = 5{,}760{,}000,\ \sqrt[5]{R} = 22.5$
$C_F = .074/22.5 = .0033;\ F = C_F qS = .0033 \times 6400/840 \times 8$
$= .20$ lb.

Pr. 12. $R' = 2R;\ C_F' = \sqrt[5]{2}C_F;\ F' = 2\sqrt[5]{2};\ F = 2.30;\ F = .46$ lb.

Chapter IX

Pr. 1. $C_D = \dfrac{D}{qS} = \dfrac{840 \times 11}{220^2 \times 1} = .19$

Pr. 2. The drag per foot will be double, or 1.2 lb.

Pr. 3. $P = VD = V \cdot V^2\rho/2 \cdot SC_D$
$L = V^2\rho/2 \cdot SC_L =$ constant; thus C_L varies as $1/V^2$, and P varies as $C_D/C_L^{3/2}$

Pr. 4. $C_L = \dfrac{L}{qS} = \dfrac{840 \times 6}{146.7^2 \times 1} = .235$

Pr. 5. Moment about leading edge = $M = C_m qSc$
Normal force = $N = C_n qS$
Arm = $M/N = C_m/C_n \cdot c$, Q. E. D.

Pr. 6. $C_{Di} = C_L^2/\pi \cdot S/b^2 = .64/6\pi = .034$
$C_{Di}' = .64/4\pi = .051$
$C_D = .1 - .034 + .051 = .117$

Pr. 7. $1.0d^2 = .07D^2$; $D = 3.8d$

Pr. 8. $q = L/C_L S = 2{,}000/(1.4 \times 400) = 3.57$ lb./ft.^{2}
$V^2 = 3.57 \times 840$; $V = 55$ ft./sec.

Pr. 9. C_{Di} varies as C_L^2; C_L varies as $1/V^2$; hence C_{Di} varies as $1/V^4$, Q. E. D.

Pr. 10. $\eta = \dfrac{2}{1 + \sqrt{1+1}} = .83$

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